Assessment Schedule – 2013

Scholarship Calculus (93202)

Evidence Statement

Question One

(a) Solving $\frac{dy}{dx} = \sqrt{\varphi}\frac{2e^{-2x} - e^{-x}}{2\sqrt{e^{-x} - e^{-2x}}} = \sqrt{\varphi}\frac{2e^{-2x} - e^{-x}}{2y} = 0$, we find

$$\begin{aligned} 2e^{-2x} - e^{-x} &= 0 \\ \ln 2 - 2x &= -x \\ x &= \ln 2 \end{aligned}$$

The drop is widest at $x = \ln 2 \approx 0.6931$, and then $y = \sqrt{\varphi}\sqrt{e^{-\ln 2} - e^{-2\ln 2}} = \sqrt{\varphi}\sqrt{\frac{1}{2} - \frac{1}{4}} = \frac{\sqrt{\varphi}}{2} \approx 0.6360$. It is widest $\ln 2$ cm from the rounded end **C**, and is exactly $\sqrt{\varphi}$ cm wide there.

(b) Now we need $\frac{d^2y}{dx^2} = 0$, so $e^{2x} - 6e^x + 4 = 0$.

-Solving as a quadratic in e^x we find $e^x = \frac{-(-6) \pm \sqrt{20}}{2} = 3 \pm \sqrt{5}$, so $x = \ln\left(3 \pm \sqrt{5}\right)$.

Since $0 < 3 - \sqrt{5} < 1$, $x = \ln\left(3 - \sqrt{5}\right) < 0$ and so this point is not valid, and the only turning point is at $x = \ln\left(3 + \sqrt{5}\right)$

(c)

$$\begin{aligned} \int_0^{\ln p} \pi y^2\,dx &= \pi\varphi \int_0^{\ln p} \left(e^{-x} - e^{-2x}\right)dx \\ &= \pi\varphi\left[\tfrac{1}{2}e^{-2x} - e^{-x}\right]_0^{\ln p} \\ &= \pi\varphi\left(\tfrac{1}{2} + \left(\tfrac{1}{2}e^{-2\ln p} - e^{-\ln p}\right)\right) \\ &= \pi\varphi\left(\tfrac{1}{2} + \tfrac{1}{2}\tfrac{1}{p^2} + \tfrac{1}{p}\right) \\ &= \pi\varphi\frac{p^2 - 2p + 1}{2p^2} \\ &= \pi\varphi\frac{(p-1)^2}{2p^2} \\ &= \tfrac{1}{2}\pi\varphi\left(\frac{p-1}{p}\right)^2 \end{aligned}$$

> 1(a) Expression for $\frac{dy}{dx} = 0$ [1st mark],
> widest at $x = \ln 2$ [2nd mark],
> width is $\sqrt{\varphi}$ [3rd mark].
> 1(b) Recognise quadratic $a^2 - 6a + 4 = 0$ and solve [1st mark],
> find $x = \ln\left(3 + \sqrt{5}\right)$ [2nd mark].
>
> 1(c) Definite integral $= \pi\varphi\left[\frac{1}{2}e^{-2x} - e^{-x}\right]_0^{\ln p}$ [1st mark],
> show required form $= \frac{1}{2}\pi\varphi\left(\frac{p-1}{p}\right)^2$ [2nd mark],
> explain upper limit [3rd mark].
>
> *In all questions, minor error ignored once if one single character is incorrect, inserted or omitted.*
> *Note that the 2-mark question is not always part (a).*

Since $p - 1 < p$ and $\ln p \geq 0$ for the model, we can see that $0 \leq \frac{p-1}{p} < 1$ and so $0 \leq V < \frac{1}{2}\pi\varphi$.

(As p gets larger, the volume approaches $\frac{1}{2}\pi\varphi \approx 2.54\text{ cm}^3$. In the diagram, the drop has 99.99% of the maximal volume in the model.)

> Marks in each question part are independent, with follow through marks.

Question Two

(a) The two functions are orthogonal if $\theta = \frac{\pi}{2}$, so $0 = \langle f, g \rangle_0^1$.

$$\langle f,g\rangle_0^1 = \int_0^1 (kx+1)(x+k)\,dx$$
$$= \int_0^1 \left(kx^2 + (k^2+1)x + k\right)dx$$
$$= \left[\frac{kx^3}{3} + \frac{(k^2+1)x^2}{2} + kx\right]_0^1$$
$$= \frac{k}{3} + \frac{(k^2+1)}{2} + k = 0$$
$$k^2 + \frac{8}{3}k + 1 = 0$$
$$2k = -\frac{8}{3} \pm \sqrt{\frac{64}{9} - 4} = -\frac{8}{3} \pm \sqrt{\frac{28}{9}} = -\frac{8}{3} \pm \frac{2}{3}\sqrt{7}$$
$$k = -\frac{4}{3} \pm \frac{1}{3}\sqrt{7} = \frac{-4 \pm \sqrt{7}}{3}$$

2(a) Definite integral [1st mark],
values $k = \frac{-4 \pm \sqrt{7}}{3}$ [2nd mark].
2(b) Find at least two of $\|f\|, \|g\|, \langle f,g\rangle_0^1$ [1st mark],
find $\cos\theta = \frac{7/2}{\sqrt{7}\sqrt{7}}$ [2nd mark],
and $\theta = \frac{\pi}{3}$ [3rd mark].
2(c) Either trig identity used, or first step of integration by parts [1st mark],
show that $\langle \ldots \rangle_0^1 = 0$ when $m \neq n$ [2nd mark],
test what happens when $n = m$ [3rd mark].

(b) The angle requires finding three inner products, since $\langle f,g\rangle_0^1 \neq 0$.

$$\langle f,g\rangle_0^1 = \int_0^1 (3x-4)(9x-5)\,dx = \int_0^1 \left(27x^2 - 51x + 20\right)dx = \left[9x^3 - \frac{51}{2}x^2 + 20x\right]_0^1 = 9 - \frac{51}{2} + 20 = \frac{7}{2}$$
$$\langle f,f\rangle_0^1 = \int_0^1 (3x-4)(3x-4)\,dx = \int_0^1 \left(9x^2 - 24x + 16\right)dx = \left[3x^3 - 12x^2 + 16x\right]_0^1 = 3 - 12 + 16 = 7$$
$$\langle g,g\rangle_0^1 = \int_0^1 (9x-5)(9x-5)\,dx = \int_0^1 \left(81x^2 - 90x + 25\right)dx = \left[27x^3 - 45x^2 + 25x\right]_0^1 = 27 - 45 + 25 = 7$$
$$\cos\theta = \frac{\langle f,g\rangle_0^1}{\sqrt{\langle f,f\rangle_0^1 \cdot \langle g,g\rangle_0^1}} = \frac{7}{2} \div 7 = \frac{1}{2} \qquad \text{so } \theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$$

(c) $$\int_0^{2\pi} \sin(mx)\sin(nx)\,dx = \frac{1}{2}\int_0^{2\pi} \left(\cos((m-n)x) - \cos((m+n)x)\right)dx$$
$$= \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n}\right]_0^{2\pi} = 0 \text{ if } m \neq n$$

However, if $n = m$
$$\int_0^{2\pi} \sin(mx)\sin(nx)\,dx = \frac{1}{2}\int_0^{2\pi} \left(1 - \cos(2nx)\right)dx = \frac{1}{2}\left[x - \frac{\sin(2nx)}{2n}\right]_0^{2\pi} = \pi - \frac{\sin(4n\pi)}{4n} = \pi \neq 0.$$

So $\sin(nx)$ and $\sin(mx)$ are othogonal if $m \neq n$.

(see also integration by parts in appendix)

Question Three

(a) (i) A polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is:
even if and only if the odd coefficients are all zero, and
odd if and only if the even coefficients are all zero.
neither if there are both even and odd coefficients which are non-zero.

Candidates might note that $p(x) = 0$ is the unique polynomial which is both even and odd.

[It is *insufficient* to say that $p(x) = ax^{2n}$ is even, and $p(x) = ax^{2n+1}$ is odd, unless also noting that the sum of odd polynomials is odd, and the sum of even polynomials is even.]

(ii) We are given that g is an even function, so $g(-x) = g(x)$ for all x.
We use the facts that $g(-x+h) = g(x-h)$ and $g(-x) = g(x)$.

$$\begin{aligned} g'(-x) &= \lim_{h\to 0} \frac{g(-x+h) - g(-x)}{h} \\ &= \lim_{h\to 0} \frac{g(x-h) - g(x)}{h} \\ &= \lim_{k\to 0} \frac{g(x+k) - g(x)}{-k} \quad \text{where } k = -h \\ &= -\lim_{k\to 0} \frac{g(x+k) - g(x)}{k} \\ &= -g'(x) \end{aligned}$$

Since $g'(-x) = -g'(x)$ we have shown that $\frac{dg}{dx}$ is an odd function.

3(a)(i) Recognise building block functions (powers) as odd and even [1st mark],
full description of odd, even and neither [2nd mark].
3(a)(ii) Write expression for $g'(-x)$ [1st mark],
use $g(-x+h) = g(x-h)$ [2nd mark],
full proof [3rd mark].
3(b) Find $\frac{d^2y}{dx^2}$ [1st mark],
get form $\frac{d^3y}{dx^3} = Ay + Be^{-x}\cos kx$ [2nd mark],
find $C = 8$ [3rd mark].

(b) We find the third derivative, then look to the coefficients of the terms.

$$\frac{dy}{dx} = -e^{-x}\sin(kx) + ke^{-x}\cos(kx)$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= e^{-x}\sin(kx) - ke^{-x}\cos(kx) - ke^{-x}\cos(kx) - k^2e^{-x}\sin(kx) \\ &= (1-k^2)e^{-x}\sin(kx) - 2ke^{-x}\cos(kx) \end{aligned}$$

$$\begin{aligned} \frac{d^3y}{dx^3} &= (k^2-1)e^{-x}\sin(kx) + (k-k^3)e^{-x}\cos(kx) + 2ke^{-x}\cos(kx) + 2k^2e^{-x}\sin(kx) \\ &= (3k^2-1)e^{-x}\sin(kx) + (3k-k^3)e^{-x}\cos(kx) \\ &= (3k^2-1)y + k(3-k^2)e^{-x}\cos(kx) \end{aligned}$$

To have $\frac{d^3y}{dx^3} = Cy$ we must have $k(3-k^2) = 0$ so $k = \pm\sqrt{3}$ and then $C = 3k^2 - 1 = 8$.

Question Four

(a) For each $2 \le n \le 9$ we list the solutions of $z^n = z$; these are the solutions of $z^{n-1} = 1$; roots of unity.

We need to be careful not to count any roots twice; this is easiest when writing the solutions in the form $z = \text{cis}\left(\frac{a}{n-1}\pi\right)$.

$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$z=\text{cis}\,0\pi$	$z=\text{cis}\,\pi$	$z=\text{cis}\frac{2}{3}\pi$ $z=\text{cis}\frac{4}{3}\pi$	$z=\text{cis}\frac{1}{2}\pi$ $z=\text{cis}\frac{3}{2}\pi$	$z=\text{cis}\frac{2}{5}\pi$ $z=\text{cis}\frac{4}{5}\pi$ $z=\text{cis}\frac{6}{5}\pi$ $z=\text{cis}\frac{8}{5}\pi$	$z=\text{cis}\frac{1}{3}\pi$ $z=\text{cis}\frac{5}{3}\pi$	$z=\text{cis}\frac{2}{7}\pi$ $z=\text{cis}\frac{4}{7}\pi$ $z=\text{cis}\frac{6}{7}\pi$ $z=\text{cis}\frac{8}{7}\pi$ $z=\text{cis}\frac{10}{7}\pi$ $z=\text{cis}\frac{12}{7}\pi$	$z=\text{cis}\frac{1}{4}\pi$ $z=\text{cis}\frac{3}{4}\pi$ $z=\text{cis}\frac{5}{4}\pi$ $z=\text{cis}\frac{7}{4}\pi$
1	1	2	2	4	2	6	4

There are 22 solutions in the form $z = \text{cis}\left(\frac{a}{n-1}\pi\right)$, and also the solution $z = 0$.

There are **23 solutions in total**.

(b)(i)

We rearrange to collect Δv terms together.

$$\left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}} = \frac{1+\frac{\Delta v}{c}}{1-\frac{\Delta v}{c}}$$

$$\left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}}\left(1-\frac{\Delta v}{c}\right) = 1+\frac{\Delta v}{c}$$

$$\left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}} - \left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}}\frac{\Delta v}{c} = 1+\frac{\Delta v}{c}$$

$$\left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}} - 1 = \frac{\Delta v}{c}\left(\left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}} + 1\right)$$

$$\Delta v = c\frac{\left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}} - 1}{\left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}} + 1}$$

$$= c\frac{\exp\left(2\ln\left(\frac{m_0}{m_1}\right)\frac{u}{c}\right) - 1}{\exp\left(2\ln\left(\frac{m_0}{m_1}\right)\frac{u}{c}\right) + 1}$$

$$= c\tanh\left(\frac{u}{c}\ln\left(\frac{m_0}{m_1}\right)\right)$$

Also possible to work this backwards.

4(a) Understanding of roots of unity (could be shown in a diagram) [1st mark],
find all non-zero roots (repetition allowed) [2nd mark],
23 solutions in total (allow MEI for 22) [3rd mark].
4(b)(i) Multiply out $\left(1-\frac{\Delta v}{c}\right)$ [1st mark],
get Δv as subject, any form [2nd mark],
required form [3rd mark].
OR an equivalent form in the opposite direction, with substituting $\tanh$ correctly as [1st mark].
4(b)(ii) Differentiate both sides, or integrate with partial fractions [1st mark],
into required form [2nd mark].

(b)(ii)

We differentiate the given equation and show it satisfies the differential equation.

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(\ln M\right)=-\frac{c}{2u}\frac{\mathrm{d}}{\mathrm{d}v}\left(\ln\left(\frac{1+v/c}{1-v/c}\right)\right)$$

$$\frac{1}{M}\frac{\mathrm{d}M}{\mathrm{d}v}=-\frac{c}{2u}\times\frac{\left(1-v/c\right)}{\left(1+v/c\right)}\times\frac{\frac{1}{c}\left(1-v/c\right)-\frac{-1}{c}\left(1+v/c\right)}{\left(1-v/c\right)^2}$$

$$\frac{\mathrm{d}M}{\mathrm{d}v}=\frac{-Mc}{2u}\frac{\frac{2}{c}}{\left(1+v/c\right)\left(1-v/c\right)}$$

$$=\frac{-M}{u\left(1-v^2/c^2\right)}$$

Question Five

(a) We find the critical points where $\frac{\mathrm{d}A}{\mathrm{d}\theta} = 0$:

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = -\tfrac{1}{2}R^2 + \tfrac{1}{4}\pi R^2(2\sin\theta)(1-\cos\theta) + \tfrac{1}{2}R^2\cos^2\theta - \tfrac{1}{2}R^2\sin^2\theta$$

$$0 = -1 + \pi\sin\theta(1-\cos\theta) + \cos^2\theta - \sin^2\theta$$

$$0 = -1 + \pi\sin\theta(1-\cos\theta) + 1 - \sin^2\theta - \sin^2\theta$$

$$0 = \pi\sin\theta(1-\cos\theta) - 2\sin^2\theta$$

$$0 = \sin\theta\left(\pi(1-\cos\theta) - 2\sin\theta\right)$$

$$0 = \sin\theta\left(\pi\left(-2\sin^2\tfrac{\theta}{2}\right) - 4\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\right)$$

$$0 = \sin\theta\cdot\sin\tfrac{\theta}{2}\cdot\left(\pi\sin\tfrac{\theta}{2} - 2\cos\tfrac{\theta}{2}\right)$$

So either $\sin\theta = 0$ (giving $\theta = n\pi$, which are the maxima at 0 and π), or:

$$\pi\sin\tfrac{\theta}{2} = 2\cos\tfrac{\theta}{2}$$

$$\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{2}{\pi}$$

$$\tan\tfrac{\theta}{2} = \frac{2}{\pi}$$

$$\tfrac{\theta}{2} = \tan^{-1}\left(\tfrac{2}{\pi}\right)$$

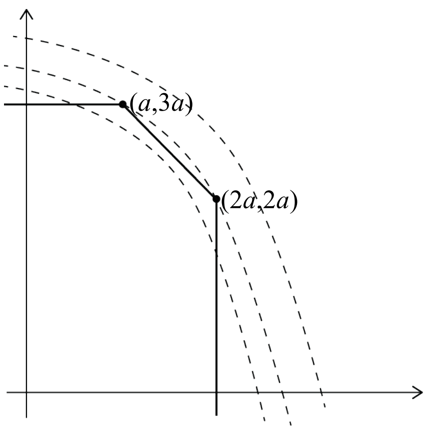
$$\theta = 2\tan^{-1}\left(\tfrac{2}{\pi}\right)$$

> 5(a) Find any form of $\frac{\mathrm{d}A}{\mathrm{d}\theta}$ [1st mark]
> take factor of $\sin\theta$ out [2nd mark],
> half-angle formula step [3rd mark],
> find solution [4th mark].

(b)

EITHER

The feasible region has vertices at $(x, y) = (2a, 2a)$ and $(x, y) = (1a, 3a)$.
The required non-linear objective function $P(x, y)$ must take its largest value at both points, like the middle line in the diagram. Larger values (the outer curve) are not met, and smaller values (the inner curve) are not maximal.

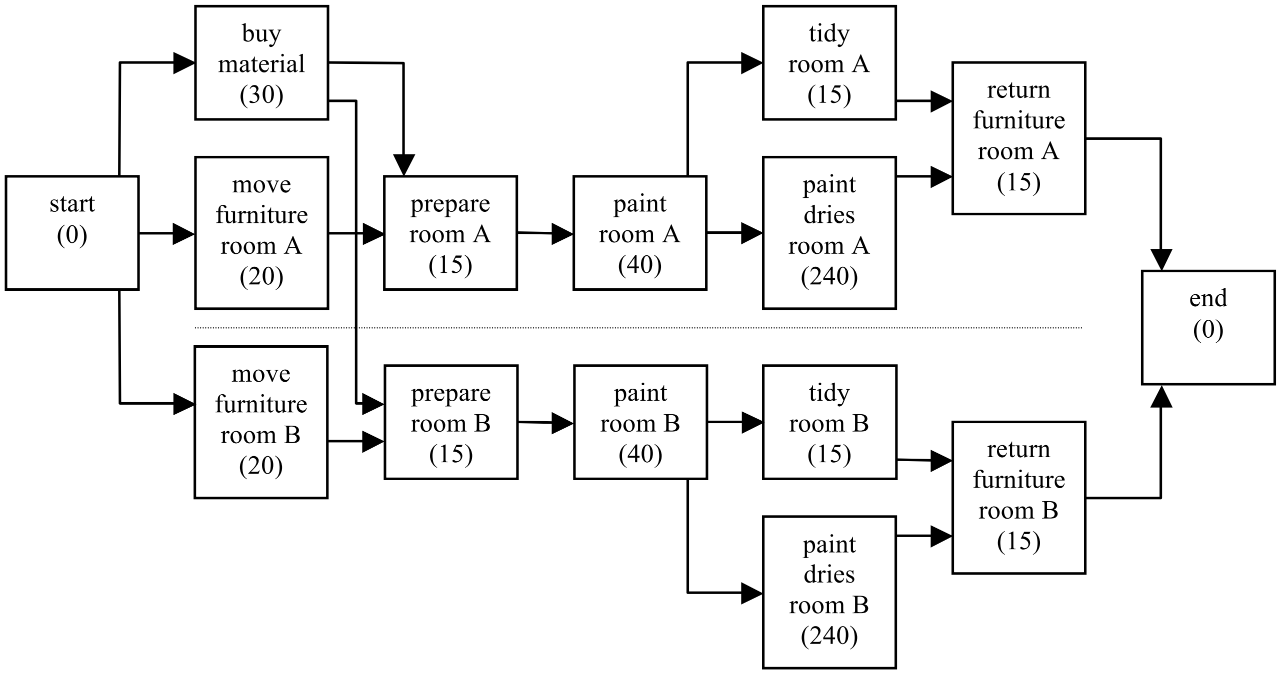


> 5(b) *linear programming*
> Find vertices $(a, 3a)$ and $(2a, 2a)$ [1st mark],
> diagram of feasible region [2nd mark],
> curved objective function through vertices [3rd mark],
> justified answer [4th mark].

OR

The critical path in the first flowchart is 420 minutes (7 hours).

When splitting the tasks out, the new flowchart would look something like this:



- The new critical path is 80 minutes shorter (340 minutes).
- This assumes the rooms are the same size, and the tasks are divided equally between the rooms.
- Splitting out the moving furniture task makes the buy material task part of the critical path. However, splitting the tidying task does not help, because the paint drying takes much longer.
- The painters might need to spend more on material if they are painting two rooms at once.
- There might not be enough painters to paint two rooms at once; or to move the furniture twice as fast.
- Other deeper-thinking statements about the new flowchart possible.

Would expect at least four good statements about the new flowchart in a full answer. Surface-level understanding not sufficient.

5(b) *critical path analysis*
Find the critical paths are 420 and 340 minutes [1st mark]
insightful mathematical statement [2nd mark]
insightful real-world statement [3rd mark]
insightful statement connecting real-world and mathematical contexts [4th mark]

Question Six

(a) Using de Moivre's Theorem, and collecting real and imaginary terms:

$$\cos 5\theta + \mathrm{i}\sin 5\theta = \operatorname{cis} 5\theta = \operatorname{cis}^5\theta$$
$$= (\cos\theta + \mathrm{i}\sin\theta)^5$$
$$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta$$
$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$\mathrm{i}\sin 5\theta = 5\mathrm{i}\cos^4\theta\sin\theta - 10\mathrm{i}\cos^2\theta\sin^3\theta + \mathrm{i}\sin^5\theta$$
$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

Alternatively, with rather more work, the sum of angle formulas can get the same results. Differentiating either identity gives the other.

> 6(a) Use de Moivre's Theorem [1st mark]
> expand (binomial theorem or otherwise [2nd mark]
> simplify, separate real / imaginary parts [3rd mark]
> well reasoned proof [4th mark].

(b)

EITHER

Using $x_{n+1} = x_n$, $y_{n+1} = y_n$ and $z_{n+1} = z_n$, we get the equations

$$x_n = 0.8x_n + 0.7y_n + 0.6z_n$$
$$y_n = 0.1x_n + 0.2y_n + 0.4z_n$$
$$z_n = 0.1x_n + 0.1y_n$$

yielding:

$$-0.2x_n + 0.7y_n + 0.6z_n = 0$$
$$0.1x_n - 0.8y_n + 0.4z_n = 0$$
$$0.1x_n + 0.1y_n - z_n = 0$$

We also need the additional information that $x_n + y_n + z_n = 99$, as the first three are insufficient.

Using any of several ways to solve this system of four equations, we find that

$x_n = 76$

$y_n = 14$

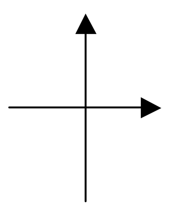
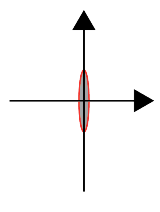
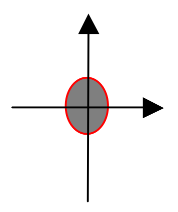
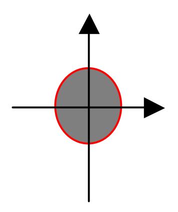
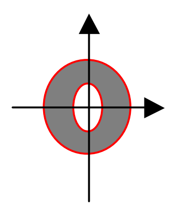
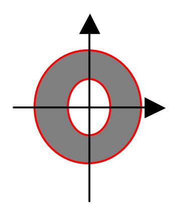
$z_n = 9$

> 6(a) *linear systems*
> Set $x_{n+1} = x_n$ etc [1st mark] collect terms [2nd mark]
> introduce extra equation $x + y + z = 99$ [3rd mark]
> solution $(x, y, z) = (76, 14, 9)$ [4th mark]

OR

Consider the equation $|z+\mathrm{i}| + |z-\mathrm{i}| = A$. This is an ellipse, with foci at i and $-\mathrm{i}$, so long as $A > 2$. When $A = 2$ it is the set of points on the imaginary axis between i and $-\mathrm{i}$.

If $k < 1$ then there are no such points.
At $k = 1$ the points are on the imaginary axis between i and $-\mathrm{i}$.
If $1 < k \le 2$ the points form a solid ellipse, with foci at i and $-\mathrm{i}$.
If $k > 2$ the points lie between two ellipses with foci at i and $-\mathrm{i}$; the outer ellipse has an inner ellipse of points which do not satisfy the inequality.
(For large *k*, both ellipses are approximately circles, with the inner circle half the radius of the outer.)

$k < 1$	$k = 1.01$	$k = 1.5$	$k = 2$	$k = 2.5$	$k = 3$
					

> 6(a) *conic sections*
> Form of ellipse with foci correct [1st mark]
> ring shape [2nd mark], solid when $1 \le k \le 2$ [3rd mark],
> no solutions when $0 < k < 1$ [4th mark].