Assessment Schedule – 2014

Scholarship Calculus (93202)

Evidence Statement

Where a response to any half-question uses a novel or unexpected solution, markers use their professional judgement and guidance from the Panel Leader to apply the following:

Across all half-questions	
0	No attempt, or little progress made
1	An attempt is made, showing an appropriate approach, but below Scholarship level
2	Significant progress is made toward an answer, at borderline Scholarship level
3	An answer is found, containing a small error, partially incomplete, or lacking clarity/insight
4	A full, clear and/or insightful solution is found – possibly an unexpected/novel solution

Combining part-question scores:

+	0	1	2	3	4
0	0	1	2	4	5
1	1	2	4	5	6
2	2	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	8

QUESTION ONE (8 marks)

(a) Note that $(7+4\sqrt{3})(7-4\sqrt{3})=1$.

Let $A=7+4\sqrt{3}$ and $k=t^2-5t+5$.

$$A^k+\left(\frac{1}{A}\right)^k=14$$

$$A^k+A^{-k}-14=0$$

$$A^{2k}-14A^k+1=0$$

$$x^2-14x+1=0$$

$$x=\frac{14\pm\sqrt{196-4}}{2}=\frac{14\pm8\sqrt{3}}{2}=7\pm4\sqrt{3}$$

So now $x=A^k=(7+4\sqrt{3})^k=7\pm4\sqrt{3}$ so $k=\pm1$.

For $k=1$, $t^2-5t+5=1$ gives $t^2-5t+4=0$ so $t=1$ or $t=4$.

For $k=-1$, $t^2-5t+5=-1$ gives $t^2-5t+6=0$ so $t=2$ or $t=3$.

The full solution set is $t\in\{1,2,3,4\}$.

QUESTION ONE (a)	
0	No attempt, or little progress made
1	One solution found
2	Two solutions found
3	Three solutions found
4	Four solutions found
Note: deduct one mark for each and every incorrect answer given	

(b) *Answer ONE of the following options.*

● **EITHER**

Let $u = \ln(\sin^{-1} e^x)$. Then $u^5 = u$.

So $u(u^4 - 1) = 0$, giving $u = 0$ or $u = \pm 1$ (ruling out the complex roots $u = \pm \mathrm{i}$).

<u>Case 1:</u> $u = 0$.

$$\ln(\sin^{-1} e^x) = 0$$
$$\sin^{-1} e^x = 1$$
$$e^x = \sin 1$$

Solution is $x = \ln(\sin 1)$.

<u>Case 2:</u> $u = 1$.

$$\ln(\sin^{-1} e^x) = 1$$
$$\sin^{-1} e^x = e$$
$$e^x = \sin(e)$$

It appears another solution is $x = \ln(\sin e)$. However, since $\sin^{-1}\left(e^{\ln(\sin e)}\right) = \sin^{-1}(\sin e) = \pi - e$ this solution *is not valid*; it does not give e as required.

<u>Case 3:</u> $u = -1$.

$$\ln(\sin^{-1} e^x) = -1$$
$$\sin^{-1} e^x = \frac{1}{e}$$
$$e^x = \sin\frac{1}{e}$$

Solution is $x = \ln(\sin\frac{1}{e})$.

There are **<u>two</u>** solutions: $x_1 = \ln(\sin 1)$ and $x_2 = \ln\left(\sin\frac{1}{e}\right)$.

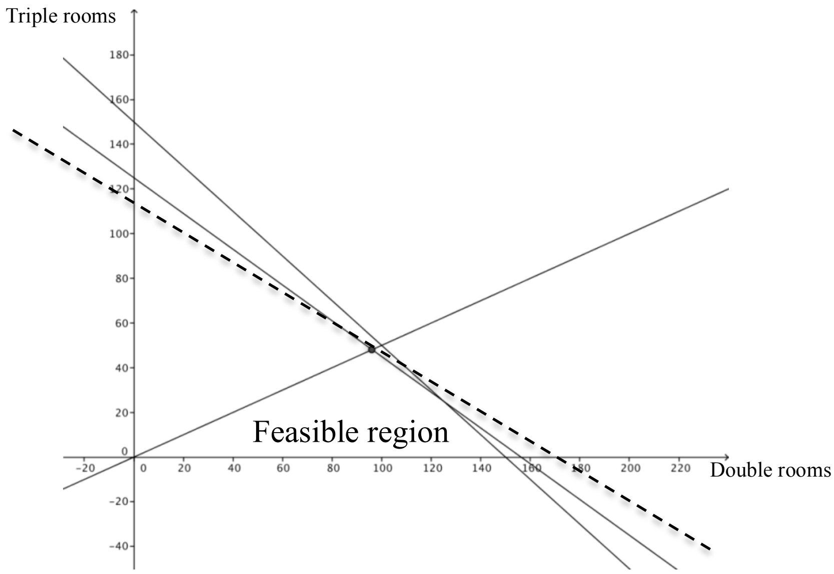
$$\ln\sin 1 \approx -0.1726$$
$$\ln\sin e \approx -0.8897$$
$$\ln\sin\frac{1}{e} \approx -1.0227$$

<table>
<tr><td colspan="3">QUESTION ONE (b) [first option]</td></tr>
<tr><td>0</td><td colspan="2">No attempt, or little progress made</td></tr>
<tr><td>1</td><td>One (potential) solution found, any method</td><td rowspan="2">numerical solutions sufficient</td></tr>
<tr><td>2</td><td>Two (potential) solutions found</td></tr>
<tr><td>3</td><td colspan="2">All three potential solutions found, exact form</td></tr>
<tr><td>4</td><td colspan="2">False solution eliminated with justification</td></tr>
</table>

- **OR**

Constraints:
$x \geq 0$ and $y \geq 0$
$x + y \leq 150$
$6400x + 800y \leq 1000000$ or equivalent $8x + 10y \leq 1250$
$3\times 2x \geq 4\times 3y$ or equivalent $x \geq 2y$

Daily Income $= 200x + 1800y$



At (96,48) the income is \$201 600 (96 double rooms and 48 triple rooms).

This uses only 144 of the available 150 rooms.

Resolutions:

(i) If they could find an extra \$40 000, the profit would be maximized at 100 double and 50 triple rooms with an increase in revenue of \$8400.
(ii) Change the charges so that a double room charge is \$1400 and a triple \$1575. This would change the income to \$214 375 if they had 125 double rooms and 25 triple rooms. However, the change in price might change the demand.

QUESTION ONE (b) [second option]	
0	No attempt, or little progress made
1	Three constraints found (can infer $x \geq 0$ and $y \geq 0$ from shaded diagram)
2	Consistent feasible region indicated
3	Maximum found
4	Resolution of issue discussed

QUESTION TWO (8 marks)

(a) First, let $z = x^2$, then $z^2 - 2kz + q^2 = 0$.

This has roots $z = \frac{2k \pm \sqrt{4k^2 - 4q^2}}{2} = k \pm \sqrt{k^2 - q^2}$.

Case 1: $q^2 > k^2$ which arises when $|q| > |k|$. So z is complex, and x is **complex**; the roots are distinct.

Case 2: $q^2 = k^2$ which arises when $q = \pm k$. Then $z = k$.

Case 2a:

If $k < 0$ then the solutions are **complex** roots: $x = \pm\sqrt{k}$ i.

Case 2b:

If $q = k = 0$ then $z = 0$ is the only root (a **repeated** root).

Case 2c:

If $k > 0$ then the solutions are real roots: $x = \pm\sqrt{k}$.

Case 3: $q^2 < k^2$ which arises when $|q| < |k|$. Then z is real.

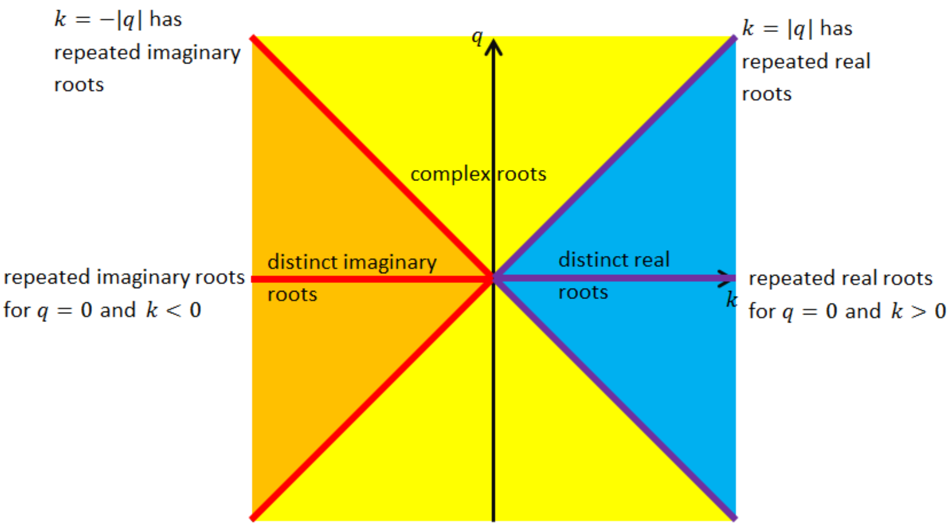
Case 3a:

If $k < 0$ then both roots are negative since $\sqrt{k^2 - q^2} < k$. So the four roots are **complex** and distinct.

Case 3b:

If $k > 0$ then both roots are positive. So the four roots are real and distinct unless $q = 0$, and then there is a **repeated** root of $x = 0$.

Diagrammatically (not required):



<table>
<tr><td colspan="3">QUESTION TWO (a)</td></tr>
<tr><td>0</td><td colspan="2">No attempt, or little progress made</td></tr>
<tr><td>1</td><td>Identifies the roots as $x = \pm\sqrt{k \pm \sqrt{k^2 - q^2}}$ in some way</td><td rowspan="3">independent marks</td></tr>
<tr><td rowspan="2">2,3</td><td>Identifies some repeated roots ($q = 0$ or $q = \pm k$)</td></tr>
<tr><td>Identifies some complex roots</td></tr>
<tr><td>4</td><td>Full discussion covering all cases</td><td></td></tr>
</table>

(b) *Answer ONE of the following options.*

- **EITHER**

We need to find the intersection of the curves $y = 9\operatorname{cosec}^2 x$ and $y = 16\sin^2 x$.

$$16\sin^2 x = 9\operatorname{cosec}^2 x$$

$$\sin^4 x = \frac{9}{16}$$

$$\sin x = \pm\frac{\sqrt{3}}{2}$$

From the graph it is clear that $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$ are the intersections.

Now, note that $16\sin^2 x = 8 - 8\cos 2x$, which allows us to find the integrals.

$\int 9\operatorname{cosec}^2 x\,dx = -9\cot x + C$

$\int 16\sin^2 x\,dx = 8x - 4\sin 2x + C$ using a trigonometric identity on the integrand.

$$A = C = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (9\operatorname{cosec}^2 x - 16\sin^2 x)dx = 6\sqrt{3} - \frac{4\pi}{3}$$

$$B = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} (16\sin^2 x - 9\operatorname{cosec}^2 x)dx = \frac{8\pi}{3} - 2\sqrt{3}$$

QUESTION TWO (b) [first option]		
0	No attempt, or little progress made	
1,2,3	Finds $\int 9\operatorname{cosec}^2 x\,dx = -9\cot x$	Marks independent
	Locates the limits of integration $x = \frac{\pi}{3}, \frac{2\pi}{3}$	
	Finds $\int 16\sin^2 x\,dx = 8x - 4\sin 2x$	
4	Finds the areas of the regions	

- **OR**

We write the equations as an augmented matrix.

$$\begin{bmatrix} c+2 & 0 & c-2 & 0 \\ c+3 & c-3 & 0 & 0 \\ 0 & c+5 & c-5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{c-2}{c+2} & 0 \\ c+3 & c-3 & 0 & 0 \\ 0 & c+5 & c-5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{c-2}{c+2} & 0 \\ 0 & c-3 & \frac{-(c+3)(c-2)}{c+2} & 0 \\ 0 & c+5 & c-5 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & \frac{c-2}{c+2} & 0 \\ 0 & 1 & \frac{-(c+3)(c-2)}{(c+2)(c-3)} & 0 \\ 0 & c+5 & c-5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{c-2}{c+2} & 0 \\ 0 & 1 & \frac{-(c+3)(c-2)}{(c+2)(c-3)} & 0 \\ 0 & 0 & c-5+\frac{(c+5)(c+3)(c-2)}{(c+2)(c-3)} & 0 \end{bmatrix}$$

The case we need to consider in detail is when $c-5+\frac{(c+5)(c+3)(c-2)}{(c+2)(c-3)}=0$.

$$\frac{(c+5)(c+3)(c-2)}{(c+2)(c-3)} = -(c-5)$$
$$(c+5)(c+3)(c-2) = -(c-5)(c+2)(c-3)$$
$$c^3+6c^2-c-30 = -c^3+6c^2+c-30$$
$$2c^3-2c=0$$
$$c(c-1)(c+1)=0$$
$$c \in \{-1,0,1\}$$

Candidates with knowledge of matrix determinants could find

$$\begin{vmatrix} c+2 & 0 & c-2 \\ c+3 & c-3 & 0 \\ 0 & c+5 & c-5 \end{vmatrix} = 2c^3-2c$$

with a similar amount of work, and note that when this determinant is zero, the system does not have a unique solution.

Since the system of equations always has the solution $(x,y,z)=(0,0,0)$ <u>the system has this unique solution when $c \notin \{-1,0,1\}$, and has infinitely many solutions (on a line) otherwise</u>.

QUESTION TWO (b) [second option]	
0	No attempt, or little progress made
1	Identifies that $(0,0,0)$ is a solution, possibly for just a particular case
2	Rearrange to find any expression involving c <u>only</u>
3	Finds the key values $c \in \{-1,0,1\}$
4	Describes how the solutions are different for these values

QUESTION THREE (8 marks)

(a) We need to find the given integral, and solve to find the value(s) of c which make it zero.

$$\int_0^{12} \rho(x)(x-c)\,\mathrm{d}x = 0$$

$$\int_0^{12} bx^r(12-x)(x-c)\,\mathrm{d}x = 0$$

$$b\int_0^{12} \left((12+c)x^{r+1} - 12cx^r - x^{r+2}\right)\mathrm{d}x = 0$$

$$\left[\frac{(12+c)x^{r+2}}{r+2} - \frac{12cx^{r+1}}{r+1} - \frac{x^{r+3}}{r+3}\right]_0^{12} = 0$$

$$\frac{(12+c)12^{r+2}}{r+2} - \frac{12c\times 12^{r+1}}{r+1} - \frac{12^{r+3}}{r+3} = 0$$

$$\frac{12(12+c)}{r+2} - \frac{12c}{r+1} - \frac{144}{r+3} = 0$$

$$\ldots$$

$$c = \frac{12(r+1)}{(r+3)}$$

QUESTION THREE (a)	
0	No attempt, or little progress made
1	Correctly expands integrand
2	Integrates correctly
3	Simplifies expressions in definite integral (eliminates $b12^r$)
4	Solves for c in terms of r

(b) *Answer ONE of the following options.*

- **EITHER**

The maxima and minima of $h_p(x)$ occur at the same places as the maxima and minima of $f(x)$ and $g(x)$; these are 1 and 3, and 25 and 27 respectively.

So the maximum of $h_p(x)$ is $3^{1-p}27^p$ and the minimum is $1^{1-p}25^p$.

Therefore $S(p) = 3^{1-p}27^p - 25^p$.

$$\frac{dS}{dp} = -\ln 3 \times 3^{1-p}27^p + \ln 27 \times 3^{1-p}27^p - \ln 25 \times 25^p = 0$$

$$3^{1-p}27^p(\ln 27 - \ln 3) = \ln 25 \times 25^p$$

$$\ln 25 \ \times 25^p = 3 \times (\ln 27 - \ln 3)\frac{1}{3^p}27^p$$

$$\frac{3^p \times 25^p}{27^p} = \frac{3(3\ln 3 - \ln 3)}{\ln 25}$$

$$\left(\frac{25}{9}\right)^p = \frac{6\ln 3}{2\ln 5} = \frac{3\ln 3}{\ln 5}$$

$$p = \log_{\frac{25}{9}}\left(\frac{3\ln 3}{\ln 5}\right) = \frac{\ln\left(\frac{3\ln 3}{\ln 5}\right)}{\ln\frac{25}{9}} = \frac{\ln\left(\frac{\ln 27}{\ln 5}\right)}{\ln\frac{25}{9}} \approx 0.702$$

Several forms are given; any <u>exact</u> form is sufficient.

Candidates might simplify $S(p) = 3^{1+2p} - 25^p$ at the start.

QUESTION THREE (b) [first option]	
0	No attempt, or little progress made
1	Finds maximum and minimum values of $f(x)$ and $g(x)$; or of $h_p(x)$
2	Finds expression for $S(p)$
3	Finds $\frac{dS}{dp}$
4	Solves to find p for maximum (any exact form)

- **OR**

Critical Path Tasks A –B – D – E – G – I – J – K, Duration 43 weeks (see diagram below)

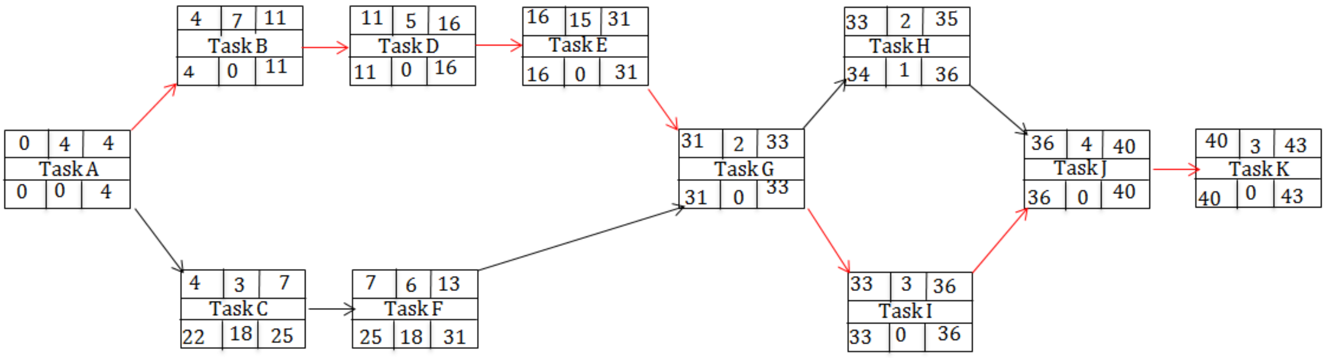
Crash the **least expensive activities on the critical path**:

Crash activity J from 4 weeks to 2 weeks	$2,400
Crash activity I from 3 weeks to 2 weeks	$2,000
Crash activity K from 3 weeks to 2 weeks	$2,000
Crash activity B from 7 weeks to 6 weeks	$2,500
Most economical cost to the clients is	**$8,900**

There are now 2 critical paths:

Tasks A –B – D – E – G – I – J – K

Tasks A –B – D – E – G – H– J – K



QUESTION THREE (b) [second option]	
0	No attempt, or little progress made
1	Identifies critical path and 43 week duration
2	Finds *any* way to crash to 38 weeks
3	Finds the most economical cost to clients
4	Discusses solution (two new critical paths)

QUESTION FOUR (8 marks)

(a) If $\operatorname{cis}(x^2) = \operatorname{cis} x$ then $x^2 = x + 2k\pi$ where k is an integer.

When $k = 0$, $x^2 = x$ so $x = 0$ or $x = 1$.

In general, $x^2 - x - 2k\pi = 0$ has roots at $x = \frac{1 \pm \sqrt{1+8k\pi}}{2}$.

Note that when $k < 0$, these will be complex arguments.

At $k = 0$ $\quad x_1 = 0 \qquad x_2 = 1$

At $k = 1$ $\quad x_3 = \frac{1}{2} - \frac{\sqrt{1+8\pi}}{2} \approx -2.06 \qquad x_4 = \frac{1}{2} + \frac{\sqrt{1+8\pi}}{2} \approx 3.06$

At $k = 2$ $\quad x_5 = \frac{1}{2} - \frac{\sqrt{1+16\pi}}{2} \approx -3.08 \qquad$ **but not** $\frac{1}{2} + \frac{\sqrt{1+16\pi}}{2} \approx 4.08 > \pi$

All other solutions with $k > 2$ are outside the required domain.

QUESTION FOUR (a)	
0	No attempt, or little progress made
1	Identifies the straightforward solutions at $x = 0$ and $x = 1$
2	Makes adjustment of $+2\pi$ or $+2k\pi$
3	Identifies two more exact solutions
4	Identifies all solutions in the domain, eliminating others

(b) *Answer ONE of the following options.*

- **EITHER**

Using the given rule, the rule for the second derivative is

$$\begin{aligned}(uvw)'' &= u''vw + u'v'w + u'vw' + u'v'w + uv''w + uv'w' + u'vw' + uv'w' + uvw'' \\ &= u''vw + uv''w + uvw'' + 2u'v'w + 2uv'w' + 2u'vw'\end{aligned}$$

Finding the full form of the solution can be shortened by the symmetry of the problem; the coefficients in each row are equal.

$$\begin{aligned}(uvw)''' = {} & 1u'''vw + 1uv'''w + 1uvw''' \\ & + 3u''v'w + 3u''vw' + 3u'v''w + 3uv''w' + 3u'vw'' + 3uv'w'' \\ & + 6u'v'w'\end{aligned}$$

$$A = B = C = 1$$

$$D = E = F = G = H = I = 3$$

$$J = 6$$

QUESTION FOUR (b) [first option]	
0	No attempt, or little progress made
1	Second derivative found, any form
2	Third derivative found, any form
3	Third derivative in correct form, stating coefficients
4	Work showing insight, clarity or flair

- **OR**

Using the definition of the binomial coefficients and rearranging the factorials:

$$\frac{n!}{(n-r)!\,r!} = \frac{(n+1)!}{\big(n+1-(r-1)\big)!\,(r-1)!}$$
$$\frac{n!}{(n-r)!\,r!} = \frac{(n+1)!}{(n-r+2)!\,(r-1)!}$$
$$\frac{(n-r+2)!}{(n-r)!} = \frac{(n+1)!\,r!}{n!\,(r-1)!}$$

Most of the factorials cancel:

$$(n-r+2)(n-r+1) = (n+1)r$$
$$n^2 - nr + n - nr + r^2 - r + 2n - 2r + 2 = nr + r$$
$$n^2 - 3nr + 3n + r^2 - 4r + 2 = 0$$
$$n^2 + (3-3r)n + (r^2 - 4r + 2) = 0$$

As a quadratic in n, the roots are:

$$n = \frac{3r - 3 \pm \sqrt{(3-3r)^2 - 4(r^2 - 4r + 2)}}{2}$$
$$n = \frac{3r - 3 \pm \sqrt{9 - 18r + 9r^2 - 4r^2 + 16r - 8}}{2}$$
$$n = \frac{3r - 3 + \sqrt{5r^2 - 2r + 1}}{2}$$

(as only the larger root is useful).

Trying some values of r, most give irrational values for n

$r = 1$ $\qquad$ $n = 1$

For $1 < r < 6$ we find n is not an integer.

$r = 6$ $\qquad$ $n = 14$

So $\binom{14}{6} = \binom{15}{5} = 3003.$

Note that searching for an integer value of $\sqrt{5r^2 - 2r + 1}$ by solving $5r^2 - 2r + 1 = k^2$ for various values of k would need a search to $k = 13$.

Other – much larger but acceptable – integer solutions exist.

QUESTION FOUR (b) [second option]	
0	No attempt, or little progress made
1	Simplify expressions for coefficients to eliminate all factorials
2	Write as a quadratic in n, or collected the n^2 and n terms
3	Find n in terms of r (with $\pm$ acceptable)
4	Find particular solution, eg $\binom{14}{6} = \binom{15}{5}$ or as $n = 14$ and $r = 6$

QUESTION FIVE (8 marks)

(a) Finding the first derivative, then rearranging the second derivative to contain only y:

$$\frac{dy}{dx} = e^{e^{cx}} \cdot e^{cx} \cdot c$$

$$\frac{d^2y}{dx^2} = e^{e^{cx}} \cdot e^{cx} \cdot c \cdot e^{cx} \cdot c + e^{e^{cx}} \cdot e^{cx} \cdot c \cdot c$$

$$\frac{d^2y}{dx^2} = e^{e^{cx}} \cdot e^{cx} \cdot c^2 \cdot (e^{cx} + 1)$$

$$\frac{d^2y}{dx^2} = c^2 \cdot y \cdot \ln y \cdot (1 + \ln y)$$

Or, with logarithms

$$\ln y = e^{cx}$$

$$\frac{1}{y}\frac{dy}{dx} = ce^{cx}$$

$$\frac{dy}{dx} = y \cdot ce^{cx}$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}ce^{cx} + yc^2e^{cx}$$

$$= yc^2e^{cx}e^{cx} + yc^2e^{cx}$$

$$= c^2 \cdot y \cdot e^{cx} \cdot (1 + e^{cx})$$

$$= c^2 \cdot y \cdot \ln y \cdot (1 + \ln y)$$

QUESTION FIVE (a)	
0	No attempt, or little progress made
1	Find the first derivative; any form
2	Find the second derivative; any form
3	Substitute to eliminate $e^{e^{cx}} = y$ (or reach this step via second form)
4	Factorise to required form

(b) *Answer ONE of the following options.*

- **EITHER**

The function is only defined when $1 - x^2 - y^2 \geq 0$. This is the region inside a circle of radius 1, centred at (0,0).

The boundaries of the rectangle arise from the other terms:

$$(x^2 - A)\big(y^2 - (1 - A)\big) = \big(x + \sqrt{A}\big)\big(x - \sqrt{A}\big)\big(y + \sqrt{1 - A}\big)\big(y - \sqrt{1 - A}\big)$$

The four lines are $x = \pm\sqrt{A}$ and $y = \pm\sqrt{1 - A}$. The corners are $\big(\pm\sqrt{A}, \pm\sqrt{1 - A}\big)$.

Since $\big(\sqrt{A}\big)^2 + \big(\sqrt{1 - A}\big)^2 = A + 1 - A = 1$ the corners lie on the circle.

The area of the circle is π.

The area of the rectangle is $R = 4\sqrt{A}\sqrt{1 - A}$.

$$4\sqrt{A(1 - A)} = \frac{\pi}{2}$$

$$\sqrt{A(1 - A)} = \frac{\pi}{8}$$

$$A(1 - A) = \frac{\pi^2}{64}$$

$$A^2 - A + \frac{\pi^2}{64} = 0$$

$$A = \frac{1 \pm \sqrt{1 - \frac{\pi^2}{16}}}{2}$$

$A \approx 0.1905, 0.8095$.

QUESTION FIVE (b) [first option]	
0	No attempt, or little progress made
1	Describes the circle and horizontal/vertical lines arising from the boundary
2	Expression for rectangle area in terms of A, $R = 4\sqrt{A}\sqrt{1 - A}$, any form
3	Find $A = \frac{1 \pm \sqrt{1 - \frac{\pi^2}{16}}}{2}$, any exact form
4	Show that the corners of the rectangle are on the circle
Note that there was a mistake in the caption of Figure 4. *The shaded regions in the diagram where $F(x, y)$ is negative.* *Candidates who have indicate they have noticed this error will be have their responses checked by the Panel Leader.*	

- **OR**

First, note we have $y = \frac{1}{x}$.

Suppose a is a variable. We aim to find b in terms of a so that the ellipse intersects the hyperbolas with a common tangent line.

This means that there is a repeated root in the solution for x in terms of a.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{a^2} + \frac{1}{b^2x^2} = 1$$

$$x^4 + \frac{a^2}{b^2} - a^2x^2 = 0$$

Let $u = x^2$:

$$u^2 - a^2u + \frac{a^2}{b^2} = 0$$

This has repeated roots when $a^4 - \frac{4a^2}{b^2} = 0$.

$$a^2\left(a^2 - \frac{4}{b^2}\right) = 0$$

So $b = \frac{2}{a}$.

Now the area of the ellipse is $A = \pi ab = \pi a \times \frac{2}{a} = 2\pi$.

Since the area of the ellipse does not vary with a, all ellipses which touch the hyperbolas have the same maximal area.

QUESTION FIVE (b) [second option]	
0	No attempt, or little progress made
1	Find an expression for the intersection in terms of x (or y) only
2	Recognise that at maximum, the intersection is a repeated root (determinant equal to zero)
3	Find $b = \frac{2}{a}$
4	Recognise that **ALL** ellipses which touch the hyperbolas have the maximal area 2π
Note that finding the area of a <u>particular</u> ellipse tangent to the hyperbolas has area 2π is not sufficient.	