Assessment Schedule – 2015

Scholarship Calculus (93202)

Evidence

QUESTION ONE (8 marks)

(a)

$$S = 2\pi\int_1^3\left(x^3+\frac{1}{12x}\right)\sqrt{1+\left(3x^2-\frac{1}{12x^2}\right)^2}\,\mathrm{d}x$$

$$= 2\pi\int_1^3\left(x^3+\frac{1}{12x}\right)\sqrt{1+9x^4-\frac{1}{2}+\frac{1}{144x^4}}\,\mathrm{d}x$$

$$= 2\pi\int_1^3\left(x^3+\frac{1}{12x}\right)\sqrt{\left(3x^2+\frac{1}{12x^2}\right)^2}\,\mathrm{d}x$$

$$= 2\pi\int_1^3\left(x^3+\frac{1}{12x}\right)\left(3x^2+\frac{1}{12x^2}\right)\mathrm{d}x$$

$$= 2\pi\int_1^3\left(3x^5+\frac{x}{12}+\frac{x}{4}+\frac{1}{144x^3}\right)\mathrm{d}x = 2\pi\int_1^3\left(3x^5+\frac{x}{3}+\frac{1}{144x^3}\right)\mathrm{d}x$$

$$S = 2\pi\left[\frac{x^6}{2}+\frac{x^2}{6}-\frac{1}{288x^2}\right]_1^3 = 2295.5$$

(b)

$$\frac{f'(x)}{\left(f(x)\right)^3} = 1 \Rightarrow \int\frac{f'(x)}{\left(f(x)\right)^3}\,\mathrm{d}x = \int 1\,\mathrm{d}x \Rightarrow \frac{-1}{2\left(f(x)\right)^2} = x+c$$

$$f(0) = 2 \Rightarrow \frac{-1}{8} = c$$

Hence,

$$\left(f(x)\right)^2 = \frac{-4}{8x-1} = \frac{4}{1-8x}$$

$$f(x) = \pm\sqrt{\frac{4}{1-8x}}$$

$$x \neq \frac{1}{8}$$

(c)

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0.8 \times 6 - \frac{6s}{200}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{960 - 6s}{200}$$

$$200\int \frac{\mathrm{d}s}{960 - 6s} = \int \mathrm{d}t$$

$$200\frac{\ln\dfrac{960 - 6s}{A}}{-6} = t$$

$$960 - 6s = A\mathrm{e}^{\frac{6}{200}t} = A\mathrm{e}^{-0.03t}$$

$$s = \frac{960 - A\mathrm{e}^{-0.03t}}{6} = 160 - C\mathrm{e}^{-0.03t}$$

$$t = 0,\ \ s = 0.5 \times 200 = 100$$

$$\Rightarrow 100 = 160 - C\mathrm{e}^{0}$$

$$s = 160 - 60\mathrm{e}^{-0.03t}$$

$$t = ?,\ \ s = 130$$

$$\Rightarrow 130 = 160 - 60\mathrm{e}^{-0.03t}$$

$$\frac{\ln 0.5}{-0.03} = 23.1\,\text{min}$$

QUESTION TWO (8 marks)

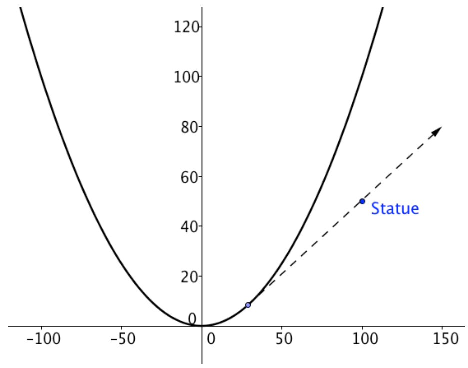
(a)

$$\left(3^{2x+y}\right)^2 - 3^{2x+y} - 6 = 0$$
$$3^{2x+y} = 3, \quad 3^{2x+y} \neq -2$$
$$2x + y = 1 \Rightarrow y = 1 - 2x$$

Substituting:

$$\log_{x+1}(4-2x)(5-x) = 3$$
$$20 + 2x^2 - 14x = (x+1)^3$$
$$x^3 + x^2 + 17x - 19 = 0$$
$$x = 1$$
$$y = -1$$

(b) The form of the parabola is $y = kx^2$.



At (100,100),

$$100 = 100^2 k \Rightarrow k = \frac{1}{100}$$
$$y = \frac{1}{100}x^2$$

Gradient of the tangent at (x_0, y_0):

$$\frac{dy}{dx} = \frac{x_0}{50}$$
$$y_0 = \frac{1}{100}x_0^2$$

Matching gradients: Gradient of tangent: $\frac{x_0}{50} = \frac{50 - \frac{1}{100}x_0^2}{100 - x_0}$ gradient of the line

$$100x_0 - x_0^2 = 2500 - \frac{1}{2}x_0^2$$
$$0 = \frac{1}{2}x_0^2 - 100x_0 + 2500$$
$$x_0 = 29.3$$

Coordinates of the point (29.3,8.6); 29.3 m east, 8.6 m north.

(c)

$$\frac{dS}{dt} = kS(N-S)$$
$$\int \frac{dS}{S(N-S)} = \int k\,dt$$

Using partial fractions:

$$\frac{1}{S(N-S)} = \frac{A}{S} + \frac{B}{N-S} \Rightarrow A = \frac{1}{N}, \quad B = \frac{1}{N}$$
$$\frac{1}{N}\int\left(\frac{1}{S} + \frac{1}{N-S}\right)dS = \int k\,dt$$

$$\frac{1}{N}\ln\frac{\frac{S}{N-S}}{A} = kt$$

$$\frac{S}{N-S} = A\mathrm{e}^{kNt}$$

$$t = 0, \quad S = 2$$

$$\Rightarrow A = \frac{2}{N-2}$$

$$\frac{S}{N-S} = \frac{2}{N-2}\mathrm{e}^{kNt} \Rightarrow S(N-2)\mathrm{e}^{-kNt} = 2(N-S)$$

$$S\left(2+(N-2)\mathrm{e}^{-kNt}\right) = 2N$$

$$S = \frac{N}{1+\frac{1}{2}(N-2)\mathrm{e}^{-kNt}}$$

OR Using differentiation:

$$S(t) = \frac{N}{1+\frac{1}{2}\mathrm{e}^{-kNt}(N-2)} = \frac{2N}{2+\mathrm{e}^{-kNt}(N-2)} = \frac{2N\mathrm{e}^{kNt}}{2\mathrm{e}^{kNt}+(N-2)}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{2kN^2\mathrm{e}^{kNt}\left(2\mathrm{e}^{kNt}+(N-2)\right)-2N\mathrm{e}^{kNt}\left(2kN\mathrm{e}^{kNt}\right)}{\left(2\mathrm{e}^{kNt}+(N-2)\right)^2}$$

$$= k\left[\frac{2N^2\mathrm{e}^{kNt}}{2\mathrm{e}^{kNt}+(N-2)} - \frac{4N^2\left(\mathrm{e}^{kNt}\right)^2}{\left(2\mathrm{e}^{kNt}+(N-2)\right)^2}\right]$$

$$= k\left(NS-S^2\right) = kS(N-S)$$

QUESTION THREE (8 marks)

(a)

$z = \cos\theta + \mathrm{i}\sin\theta, \quad z^n = \cos n\theta + \sin n\theta, \quad z^{-n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta) = \cos n\theta - \mathrm{i}\sin n\theta$

$z^n + \frac{1}{z^n} = 2\cos n\theta$

Using this

$(2\cos\theta)^6 = \left(z + \frac{1}{z}\right)^6 = z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$

$64\cos^6\theta = \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$

$= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

$32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$

$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$

(b) Restrictions: $\sin x > \frac{1}{2}, \quad y < 1$

$\log(2\sin x - 1) - \log 2 = \log(1-y) - \log(2\sin x - 1)$

$2\log(2\sin x - 1) = \log 2 + \log(1-y)$

$\log(2\sin x - 1)^2 = \log 2(1-y)$

$(2\sin x - 1)^2 = 2(1-y)$

$y = 1 - \frac{(2\sin x - 1)^2}{2}$

$\because 0 < (2\sin x - 1) \leq 1$

The range of possible values for y is

$\therefore \frac{1}{2} \leq y < 1$

(c)

$\text{LHS} = \frac{4\cos^2 2x - 4\cos^2 x + 3\sin^2 x}{4\cos^2\left(\frac{5\pi}{2} - x\right) - \sin^2 2(x-\pi)} = \frac{4\cos^2 2x - 4(\cos^2 x - \sin^2 x) - \sin^2 x}{4\sin^2 x - 4\sin^2 x\cos^2 x}$

$= \frac{4\cos^2 2x - 4\cos 2x + \frac{\cos 2x - 1}{2}}{4\sin^2 x(1-\cos^2 x)} = \frac{8\cos^2 2x - 7\cos 2x - 1}{8\sin^4 x}$

$= \frac{(8\cos 2x + 1)(\cos 2x - 1)}{8\frac{(1-\cos 2x)^2}{4}} = \frac{(8\cos 2x + 1)(\cos 2x - 1)}{2(\cos 2x - 1)^2} = \frac{8\cos 2x + 1}{2(\cos 2x - 1)} = \text{RHS}$

QUESTION FOUR (8 marks)

(a) Assume the real root exists and let it be p.

$$3p^3+(2-3ai)p^2+(6+2bi)p+4=0$$

Equating real and imaginary:

Imaginary:

$$-3ap^2+2bp=0$$

$$p(-3ap+2b)=0$$

$$p\neq 0,\quad p=\frac{2b}{3a}$$

Real:

$$3p^3+2p^2+6p+4=0$$

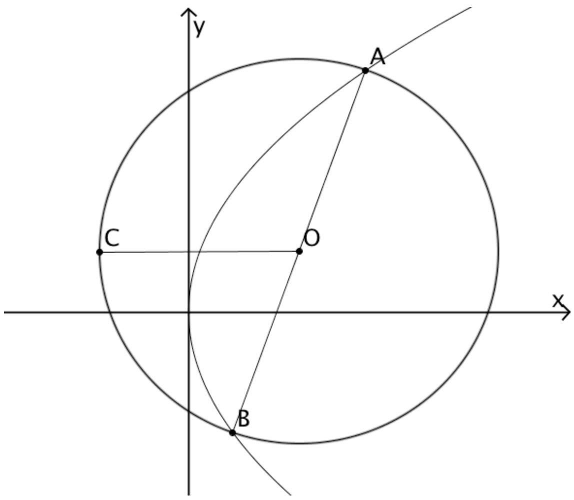
$$p^2(3p+2)+2(3p+2)=0$$

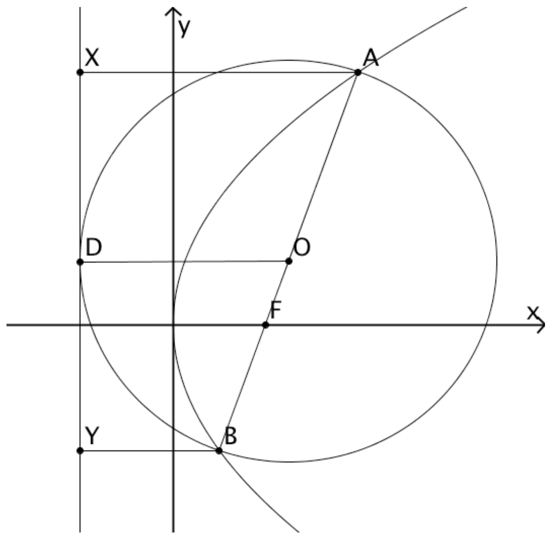
$$(3p+2)(p^2+2)=0$$

$$p^2\neq -2,\quad p=\frac{-2}{3}$$

Solution exists if $\frac{b}{a}=-1$

(b) **Option 1**





$\frac{AX+BY}{2}=OD\Rightarrow AX+BY=2OD$ Trapezium

$AX=AF$ and $BY=BF$ distance to directrix = distance to focal point

$AF+BF=2OD$

$AB=2OD$

$2AO=2OD$

$AO=OD$

(b) **Option 2**

Every hour, the maximum number of each type of decoration available is 500 type A, 400 type B and 300 type C.

$6x+2y+2z \le 500 \Rightarrow 3x+y+z \le 250 \quad (1)$

$3x+4y+5z \le 400 \quad (2)$

$2x+4y+2z \le 300 \Rightarrow x+2y+z \le 150 \quad (3)$

Every hour, the factory must pack at least 1000 decorations.

$11x+10y+9z \ge 1000 \quad (4)$

Every hour, the factory must pack more type B decorations than type A decorations.

$3x+4y+5z > 6x+2y+2z \Rightarrow 2y+3z > 3x \quad (5)$

(ii)

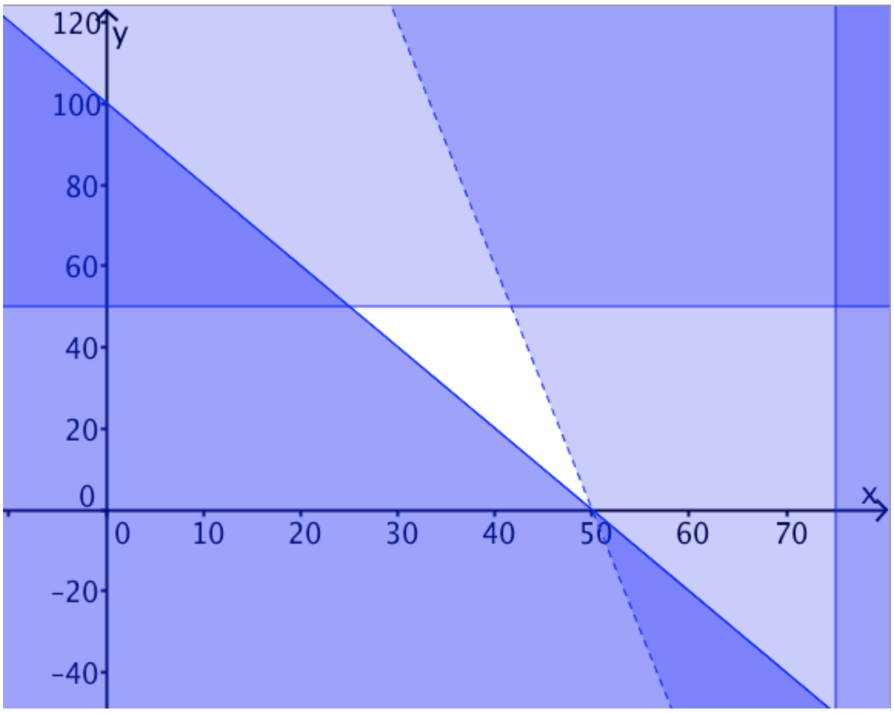
$3x+y+z \le 250 \Rightarrow 3x+y+100-x-y \le 250 \Rightarrow x \le 75 \quad (1)$

$3x+4y+5z \le 400 \Rightarrow 3x+4y+500-5x-5y \le 400 \Rightarrow 2x+y \ge 100 \quad (2)$

$x+2y+z \le 150 \Rightarrow x+2y+100-x-y \le 150 \Rightarrow y \le 50 \quad (3)$

$11x+10y+9z \ge 1000 \Rightarrow 2x+y \ge 100 \quad (4)$

$2y+3z > 3x \Rightarrow y+6x < 300 \quad (5)$



QUESTION FIVE (8 marks)

(a) Equation of the Normal

The gradient of the tangent line at the point (x_0, y_0):

$$y = kx^2$$

$$\frac{dy}{dx} = 2kx = 2kx_0$$

Gradient of the Normal line:

$$m = \frac{-1}{2kx_0}$$

Equation of the normal:

Using $y = mx + c$ at the point (x_0, y_0) to find c.

$$y_0 = \frac{-1}{2kx_0}x_0 + c \Rightarrow c = y_0 + \frac{1}{2k} \Rightarrow c = kx_0{}^2 + \frac{1}{2k}$$

$$y = \frac{-1}{2kx_0}x + \left(kx_0^2 + \frac{1}{2k}\right)$$

(b) Intersection of Normal and Parabola:

Using the general form of the normal $y = mx + c$ and $y = kx^2$.

$$kx^2 = mx + c$$

$$kx^2 - mx - c = 0$$

$$x = \frac{m \pm \sqrt{m^2 + 4kc}}{2k}$$

Substitute $m = \frac{-1}{2kx_0}$ and $c = kx_0{}^2 + \frac{1}{2k}$

$$x = \frac{m \pm \sqrt{m^2 + 4c}}{2}$$

$$x = \frac{\frac{-1}{2kx_0} \pm \sqrt{\frac{1}{4k^2x_0^2} + 4k^2x_0^2 + 2}}{2k} = \frac{\frac{-1}{2kx_0} \pm \sqrt{\left(\frac{1}{2kx_0} + 2kx_0\right)^2}}{2k} = \frac{\frac{-1}{2kx_0} \pm \left(\frac{1}{2kx_0} + 2kx_0\right)}{2k} = x_0, \quad -\left(x_0 + \frac{1}{2k^2x_0}\right)$$

To find the value of x_0 that makes $x = -\left(x_0 + \frac{1}{2k^2x_0}\right)$ a maximum (least negative):

$$\frac{dx}{d(x_0)} = -\left(1 - \frac{1}{2k^2x_0^2}\right) = 0 \Rightarrow x_0^2 = \frac{1}{2k^2} \Rightarrow x_0 = \frac{1}{\sqrt{2}k}$$

Test for maximum

$$\frac{d^2x}{d(x_0)^2} = \frac{-1}{k^2x_0^3} < 0 \text{ at } x_0 = \frac{1}{\sqrt{2}k}$$

Equation of Extreme Normal

Substituting into $y = \frac{-1}{2kx_0}x + kx_0{}^2 + \frac{1}{2k}$ gives $y = \frac{-\sqrt{2}}{2}x + \frac{1}{k}$

(c) Area

The area is an integral given as:

$$A = \int_{-x_0 - \frac{1}{2k^2 x_0}}^{x_0} \left(\frac{-1}{2kx_0} x + kx_0^2 + \frac{1}{2k} - kx^2 \right) \mathrm{d}x$$

$$A = \left[\frac{-1}{4kx_0} x^2 + \left(\frac{1}{2k} + kx_0^2 \right) x - \frac{kx^3}{3} \right]_{-x_0 - \frac{1}{2k^2 x_0}}^{x_0}$$

$$= \left[\frac{-1}{4kx_0} {x_0}^2 + \left(\frac{1}{2k} + kx_0^2 \right) x_0 - \frac{k{x_0}^3}{3} \right]$$

$$- \left[\frac{-1}{4kx_0} \left(-x_0 - \frac{1}{2k^2 x_0} \right)^2 - \left(\frac{1}{2k} + kx_0^2 \right) \left(-x_0 - \frac{1}{2k^2 x_0} \right) + \frac{k \left(-x_0 - \frac{1}{2k^2 x_0} \right)^3}{3} \right]$$

Simplifying: $A = \frac{x_0}{k} + \frac{4kx_0^3}{3} + \frac{1}{48k^5 x_0^3} + \frac{1}{4k^3 x_0}$

Differentiate to find the minimum

$$\frac{dA}{dx_0} = \frac{1}{k} + 4kx_0^2 - \frac{1}{16k^5 x_0^4} - \frac{1}{4k^3 x_0^2} = 0$$

$$= \frac{1}{16k^5 x_0^4} \left(16k^4 x_0^4 + 64k^6 x_0^6 - 1 - 4k^2 x_0^2 \right) = 0$$

$$16k^5 x_0^4 \neq 0; \quad 16k^4 x_0^4 + 64k^6 x_0^6 - 1 - 4k^2 x_0^2 = 0$$

$$16k^4 x_0^4 \left(1 + 4k^2 x_0^2 \right) - \left(1 + 4k^2 x_0^2 \right) = 0$$

$$\left(1 + 4k^2 x_0^2 \right) \left(16k^4 x_0^4 - 1 \right) = 0$$

$$1 + 4k^2 x_0^2 \neq 0; \quad 16k^4 x_0^4 - 1 = 0$$

$$x_0 = \pm \frac{1}{2k} \quad \left(x_0 > 0 \right)$$

The real positive root is $x_0 = \frac{1}{2k}$

Test minimum using the first derivative test.

Equation of the normal:

$$y = -x + \frac{3}{4k}$$

Minimum Area:

$$A_{\min} = \frac{4}{3k^2}$$