Assessment Schedule – 2016

Scholarship Calculus (93202)

Evidence Statement

Question One

(a) Method 1

$$\frac{y}{x} = \frac{\sqrt{3-(x-2)^2}}{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = \frac{-2(x-2)\cdot x}{2\sqrt{3-(x-2)^2}\cdot x^2} - \frac{\sqrt{3-(x-2)^2}}{x^2} = 0$$

$$(x-2)^2 - 3 - (x-2)x = 0; \quad x = \frac{1}{2}, \ \left(\frac{y}{x}\right)_{\max} = \sqrt{3}$$

Method 2

$$\left(\frac{y}{x}\right)^2 = \frac{3-(x-2)^2}{x^2} = m$$

$$(m+1)x^2 + 4x + 1 = 0, \Delta = 16 - 4(m+1) \geq 0, m \leq 3, \left(\frac{y}{x}\right) \leq \sqrt{3}$$

(b)(i)

$$y' = 2x\ln(1+x) + \frac{x^2}{1+x} = 2x\ln(1+x) + \left(x - 1 + (1+x)^{-1}\right)$$

$$y'' = 2\ln(1+x) + \left(\frac{2x}{1+x}\right) + 1 + (-1)(1+x)^{-2}$$

$$= 2\ln(1+x) + \left(2 - 2(1+x)^{-1}\right) + 1 + (-1)(1+x)^{-2}$$

$$f^{(2)}(0) = 0 + 0 + 1 - 1 = 0$$

(ii)

$$y' = 2x\ln(1+x) + \frac{x^2}{1+x} = 2x\ln(1+x) + \left(x - 1 + (1+x)^{-1}\right)$$

$$y'' = 2\ln(1+x) + \left(\frac{2x}{1+x}\right) + 1 + (-1)(1+x)^{-2}$$

$$= 2\ln(1+x) + \left(2 - 2(1+x)^{-1}\right) + 1 + (-1)(1+x)^{-2}$$

$$y''' = 2(1+x)^{-1} - 2(-1)(1+x)^{-2} + (-1)(-2)(1+x)^{-3}$$

$$\ldots$$

$$y^{(2016)} = 2(-1)(-2)\ldots(-2013)(1+x)^{-2014} - 2(-1)(-2)\ldots(-2014)(1+x)^{-2015} + (-1)(-2)\ldots(-2015)(1+x)^{-2016}$$

$$y^{(2016)}(0) = -2\times 2013! - 2\times 2014! - 2015!$$

$$\left(= -2013!(2015\times 2016) = \frac{-2016!}{2014} \right)$$

Question Two

(a)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sin^5 x + \cos^5 x\right) dx = 0 + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos^5 x\right) dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - \sin^2 x\right)^2 \cos x \, \mathrm{d}x$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\left(1 - 2\sin^2 x + \sin^4 x\right)\cos x\right] \mathrm{d}x = \left[\sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{16}{15}$$

(b)

$$I_n = \int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{\sin x} dx$$

$$I_n - I_{n-1} = \int_0^{\frac{\pi}{2}} \frac{\sin 2nx - \sin(2nx - 2x)}{\sin x} dx = \int_0^{\frac{\pi}{2}} \frac{2\sin x \cos(2nx - x)}{\sin x} dx$$

$$= \frac{2}{2n-1} \sin \frac{(2n-1)\pi}{2} = \frac{2(-1)^{n-1}}{2n-1}$$

(c)

Note the following facts:

$$\tan\theta + \cot\theta = \frac{1}{\sin\theta\cos\theta}$$

$$\log_a b = \frac{\log b}{\log a}.$$

$$\therefore \log_{\tan\theta + \cot\theta} \cos\theta = \frac{\log\cos\theta}{\log(\tan\theta + \cot\theta)} = \frac{\log\cos\theta}{-\log(\sin\theta\cos\theta)} = -\frac{\log\cos\theta}{\log\sin\theta + \log\cos\theta}$$

$$= -\frac{1}{\dfrac{\log\sin\theta}{\log\cos\theta} + 1} = k \quad \rightarrow \quad \frac{\log\sin\theta}{\log\cos\theta} = -\frac{1}{k} - 1$$

$$\log_{\tan\theta} \sin\theta = \frac{\log\sin\theta}{\log\tan\theta} = \frac{\log\sin\theta}{\log\sin\theta - \log\cos\theta} = \frac{1}{1 - \dfrac{\log\cos\theta}{\log\sin\theta}} = \frac{1}{1 + \dfrac{k}{k+1}} = \frac{k+1}{2k+1}$$

Question Three

(a) Let $f(x) = x - a$.

$$a = \int_0^{\frac{\pi}{2}} f(x)\sin x\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} (x-a)\sin x\,\mathrm{d}x = \left[-x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x + \left[a\cos x\right]_0^{\frac{\pi}{2}} = 1 - a$$

$$a = \frac{1}{2} \quad \therefore f(x) = x - \frac{1}{2}$$

(b)

(1) $\dfrac{d(e^{2x}y)}{dx} = 2e^{2x}y + e^{2x}\dfrac{dy}{dx} = 2e^{2x}y + e^{2x}(x-2y) = xe^{2x}$

(2) $e^{2x}y = \int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$

$$e^2 = \frac{1}{2}e^2 - \frac{1}{4}e^2 + c \rightarrow c = \frac{3}{4}e^2$$

$$y = e^{-2x}\left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + \frac{3}{4}e^2\right) = \frac{1}{4}\left(3e^{2-2x} + 2x - 1\right)$$

Question Four

(a) Asymptotes $y = \pm 3x$, $\tan\alpha = 3$, $\dfrac{\sin^2\alpha}{1-\sin^2\alpha} = 9$, $\sin^2\alpha = \dfrac{9}{10}$, $\cos^2\alpha = \dfrac{1}{10}$

$$\sin\theta = \sin(180^0 - 2\alpha) = \sin(2\alpha) = 2\times\sqrt{\frac{9}{10}}\times\sqrt{\frac{1}{10}} = 0.6$$

(b) **Method 1**

$$y = \pm 3x,\ \frac{|3x_0 - y_0|}{\sqrt{10}}\cdot\frac{|3x_0 + y_0|}{\sqrt{10}} = \frac{36}{10} = 3.6$$

Method 2

$l_1 : y = 3x,$

$$l_{PA} : y - y_0 = -\frac{1}{3}(x - x_0) \rightarrow A : (0.1x_0 + 0.3y_0,\ 0.3x_0 + 0.9y_0)$$

$$PA = \sqrt{0.9x_0^2 + 0.1y_0^2 - 0.6x_0y_0} = \frac{1}{\sqrt{10}}|3x_0 - y_0|$$

$l_2 : y = -3x,$

$$l_{PB} : y - y_0 = \frac{1}{3}(x - x_0) \rightarrow B : (0.1x_0 - 0.3y_0,\ -0.3x_0, +0.9y_0)$$

$$PB = \sqrt{0.9x_0^2 + 0.1y_0^2 + 0.6x_0y_0} = \frac{1}{\sqrt{10}}|3x_0 + y_0|$$

$$PA\cdot PB = \frac{|3x_0 - y_0|}{\sqrt{10}}\cdot\frac{|3x_0 + y_0|}{\sqrt{10}} = \frac{36}{10} = 3.6$$

(c) Let $P(x_0, y_0)$, $m_{CD} = k$. Note: $|k| > 3$

$l_{CD} : y - y_0 = k(x - x_0)$, $y = \pm 3x$

$$C : \left(\frac{kx_0 - y_0}{k-3}, \frac{3(kx_0 - y_0)}{k-3}\right),\ D : \left(\frac{kx_0 - y_0}{k+3}, \frac{-3(kx_0 - y_0)}{k+3}\right)$$

Note that $\sin\angle COD = 0.6$

$$\text{Area}_{\Delta COD} = \frac{1}{2}\times 0.6\times\sqrt{10}\left(\frac{kx_0 - y_0}{k-3}\right)\times\sqrt{10}\left(\frac{kx_0 - y_0}{k+3}\right)$$

$$= \frac{3\left(kx_0 - y_0\right)^2}{(k^2 - 9)} \qquad (1)$$

Since $CP : PD = 1 : \lambda$,

$$x_0 = \frac{1}{\lambda+1}\left(\lambda\ \frac{kx_0 - y_0}{k+3} + \frac{kx_0 - y_0}{k-3}\right) = \frac{kx_0 - y_0}{\lambda+1}\left(\frac{\lambda}{k+3} + \frac{1}{k-3}\right),$$

$$y_0 = \frac{1}{\lambda+1}\left(\lambda\ \frac{-3(kx_0 - y_0)}{k+3} + \frac{3(kx_0 - y_0)}{k-3}\right) = \frac{3(kx_0 - y_0)}{\lambda+1}\left(\frac{-\lambda}{k+3} + \frac{1}{k-3}\right) \qquad (2)$$

$$\frac{x_0^2}{4} - \frac{y_0^2}{36} = 1 \rightarrow \frac{\left(kx_0 - y_0\right)^2}{(k^2 - 9)} = \frac{\left(\lambda+1\right)^2}{\lambda}$$

$$\text{area}_{\Delta COD} = \frac{3\left(\lambda+1\right)^2}{\lambda} = 3\left(\lambda + 2 + \frac{1}{\lambda}\right),$$

min $\text{area}_{\Delta COD} = 12$, when $\lambda = 1$ (since $\lambda + \dfrac{1}{\lambda} \geq 2$).

Question Five

(a)

$$z_i = \cos\left(\frac{2k\pi}{11}\right) + i\sin\left(\frac{2k\pi}{11}\right),\ k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5,$$

$$\because z^{11} + 0 \cdot z - 1 = 0$$

$$\therefore \sum_{1}^{11} z_i = 0;$$

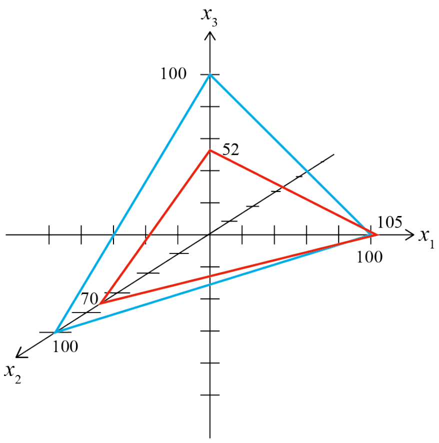
$$1 + \left(\cos\left(\pm\frac{2\pi}{11}\right) + i\sin\left(\pm\frac{2\pi}{11}\right)\right) + \ldots + \left(\cos\left(\pm\frac{10\pi}{11}\right) + i\sin\left(\pm\frac{10\pi}{11}\right)\right) = 0$$

$$\cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{4\pi}{11}\right) + \cos\left(\frac{6\pi}{11}\right) + \cos\left(\frac{8\pi}{11}\right) + \cos\left(\frac{10\pi}{11}\right) = -\frac{1}{2}$$

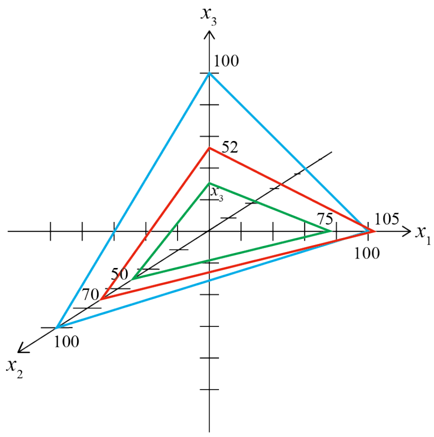
(b) ***EITHER***

	Question Type		
	Group 1 (x_1)	Group 2 (x_2)	Group 3 (x_3)
Marks	4	5	6
Time	2	3	4

$x_1 + x_2 + x_3 \le 100$ constraint 1
$2x_1 + 3x_2 + 4x_3 \le 210$ constraint 2
$2x_1 + 3x_2 \le 150$ constraint 3
$x_1 \ge 0$ constraint 4
$x_2 \ge 0$ constraint 5
$x_3 \ge 0$ constraint 6
Objective function: Grade = $4x_1 + 5x_2 + 6x_3$
Feasible Solutions are found at vertices. (0,0,0) is feasible but not helpful. Constraint (1) has vertices (100,0,0), (0,100,0) and (0,0,100). However, constraints 3 and 2 define the max values for the question groups as $x_1 \le 75$, $x_2 \le 50$ and $x_3 \le 52$.
Constraint 2 has vertices (105,0,0), (0,70,0), (0,0,52.5). This plane lies mostly between the plane defined by constraint 1 and the origin. All intersections lie in a region where $x_1 > 75$. By constraint 3, no feasible solutions in this region. Any feasible solutions now lie between plane 2 and the origin.
Only one vertex of plane 2 offers a feasible solution, being (0,0,52.5). After truncation, the objective function yields Grade = $4 \times 0 + 5 \times 0 + 6 \times 52 = 312$.
The vertices of constraint 3 are (75,0,0), (0,50,0) and (0,0,x_3): a plane with one side fixed and the other two dependent on x_3.



This plane lies between the planes defined by constraints 1 & 2 and the origin, with no intersections.
Objective function applied to the vertices (75,0,0) and (0,50,0) gives us, respectively,
Grade = $4 \times 75 = 300$ and Grade = $5 \times 50 = 250$. We have no improvement on 312.
The value of x_3 in constraint 3 is "checked" by constraint 2. Consider the boundary equations from constraints 2 & 3:
$2x_1 + 3x_2 + 4x_3 \le 210$ eq 2
$2x_1 + 3x_2 = 150$ eq 3
Eq 2 – eq 3 gives $4x_3 = 60$ or $x_3 = 15$.
So, fixing $x_3 = 15$ means the other two vertices of constraint 3 are (75,0,15) and (0,50,15).



Applying these vertices to the objective function gives:

(0,0,15) Grade = 6 × 15 = 90

(75,0,15) Grade = 75 × 4 + 15 × 6 = 390

(0,50,15) Grade = 50 × 5 + 15 × 6 = 340

The student should therefore complete 75 group 1 questions and 15 group 3.

Alternative solutions:

Iterative approach to Linear Programming Question – some students may know this method from extension classes or conjoint university papers.

Table 1

Introducing slack variables x_4, x_5 and x_6

Grades	4	5	6	0	0	0	Eqn	
Variable	x_1	x_2	x_3	x_4	x_5	x_6		
X4	1	1	1	1	0	0	100	100
X5	2	3	4	0	1	0	210	52.5
X6	2	3	0	0	0	1	150	NA
Zi	0	0	0	0	0	0		
Δi	–4	–5	-6	0	0	0		

Table 2, new basis x_4, x_3, x_6

Grades	4	5	6	0	0	0	Eqn	
Variable	x_1	x_2	x_3	x_4	x_5	x_6		
X4	1/2	1/4	0	1	–1/4	0	47.5	95
X3	1/2	3/4	1	0	1/4	0	52.5	105
X6	2	3	0	0	0	1	150	75
Zi	3	18/4	6	0	3/2	0		
Δi	–1	–1/2	0	0	3/2	0		

Table 3, new basis x_4, x_3, x_1

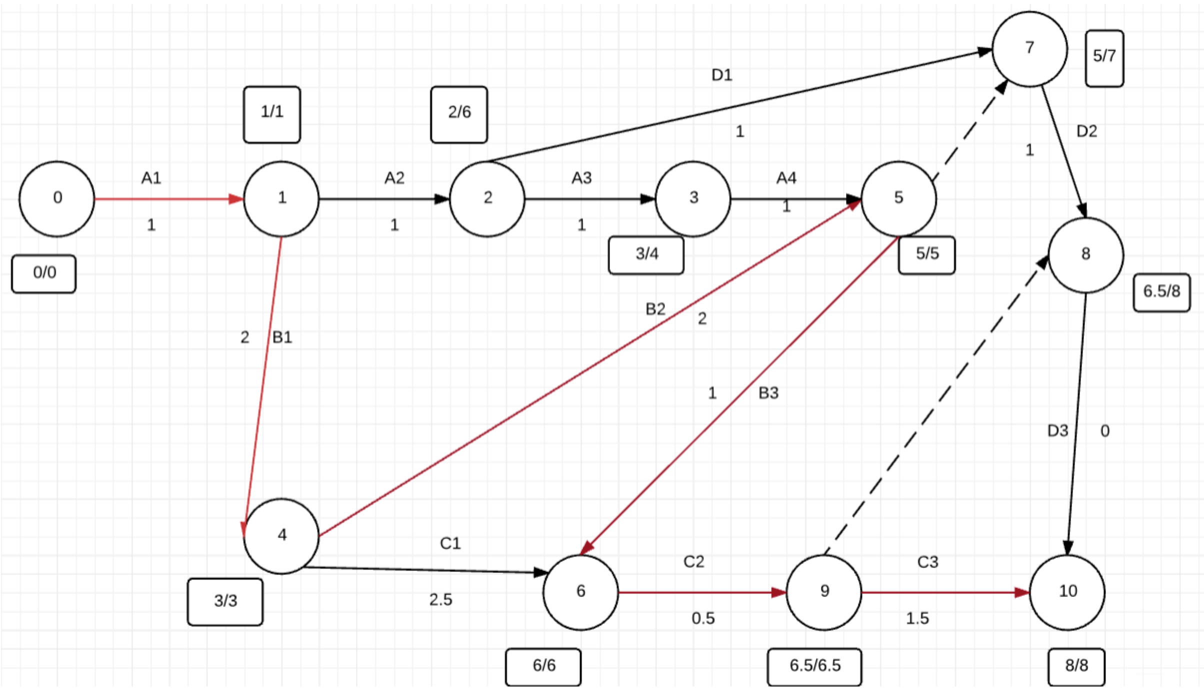
Grades	4	5	6	0	0	0	Eqn	
Variable	x_1	x_2	x_3	x_4	x_5	x_6		
X4	0	–1/2	0	1	–1/4	0	10	
X3	0	0	1	0	¼	0	15	
X1	1	3/2	0	0	0	1/2	75	
Zi	4	6	6	0	3/2	2		
Δi	0	1	0	0	3/2	2		

No negative Δi, end of iterations and the optimal basis is x_4, x_3, x_1. x_4 is slack with zero grade value, so the objective function gives Grade = $75 \times 4 + 15 \times 6 = 390$.

OR

We redefine the tasks by splitting A, B, C and D into subtasks.

Task	Duration	Dependency tree
A1 (1/3 complete)	1	
A2 (2/3 complete)	1	A1
A3 (complete)	1	A2
A4 (1 hour after completion of A)	1	A3
B1 (1/2 of B)	2	A1
B2 (1/2) of B)	2	B1, A4
B3 (1 hour after completion of B)	1	B2
C1 (first 1/2 of C)	2.5	B1
C2 (3/5 of C complete)	0.5	B3
C3 (final 2/5 of C)	2	C2
D1 (first 1/2 of D)	1	A2, B2
D2 (second 1/2 of D)	1	D1
D3 (Completion)	0	D1,C2



Associated Network : Critical Path is A1, B1, B2, B3, C2, C3. Duration is 8.5 hours.

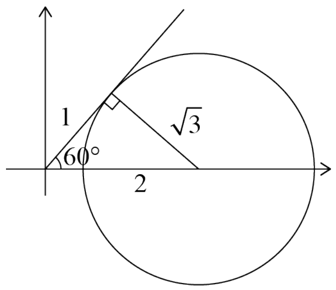
Additional Solutions Scholarship Calculus 93202

Question One

(a) (x,y) is on circle. Maximum of $\frac{y}{x} = \boldsymbol{k} \quad y = \boldsymbol{k}x$

Want maximum gradient $\boldsymbol{k}$

Occurs when $\boldsymbol{k}$ = tan(angle) = tan 60 = $\sqrt{3}$



Question Three

(a) Clearer solution.
Note that a definite integral is a constant, thus $f(x) = x - a$, $f'(x) = 1$ and $f(0) = -a$.

$$a = \int_0^{\frac{\pi}{2}} f(x)\sin x\,\mathrm{d}x = [-f(x)\cos x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} f(x)(-\cos x)\,\mathrm{d}x$$

$$a = f(0) + \int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x$$

$$a = -a + 1$$

$$a = \frac{1}{2}$$

Required function $(x) = x - \frac{1}{2}$

Question Five

(b): Alternative (shorter) method for Linear Programming option.

Further constraints: $2x_1 + 3x_2 + 4x_3 \leq 210 \quad 2x_1 + 3x_2 \leq 150 \Rightarrow x_3 \geq 15$
As the only constraint (apart from questions answered) is time related, to maximise the grade we should consider points per minute when determining what is the optimal solution.

$G = 4x_1 + 5x_2 + 6x_3$

Question	Time	Points	Points per minute
x_1	2	4	2
x_2	3	5	1.67
x_3	4	6	1.5

So complete as many x_1 as possible (150 minutes ⇒ 75 of these)
The rest of the time should be spent on x_3 (60 minutes left ⇒ 15 of these).

Maximum grade = 390 with the combination $(x_1, x_2\ x_3) = (75,0,15)$

(b) Networks option – Gantt chart solution