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NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

Scholarship 2013
Calculus

2.00 pm Monday 18 November 2013
Time allowed: Three hours
Total marks: 40

QUESTION BOOKLET

There are six questions in this booklet. Answer ANY FIVE questions.

Write your answers in Answer Booklet 93202A.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Start your answer to each question on a new page. Carefully number each question.

Check that this booklet has pages 2–8 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

You have three hours to complete this examination.

This examination consists of SIX questions.

Answer any FIVE questions.

QUESTION ONE (8 Marks)

Prince Rupert's drops are made by dripping molten glass into cold water. A typical drop is shown in Figure 1.

Figure 1: A seventeenth century drawing of a typical Prince Rupert's drop. Image from *The Art of Glass* p 354, translated and expanded from *L'Arte Vetraria* (1612) by Antonio Neri.

A mathematical model for a drop as a volume of revolution uses $y = \sqrt{\phi\left(e^{-x} - e^{-2x}\right)}$ for $x \geq 0$, and is is shown in Figure 2, where ϕ is the golden ratio $\phi = \dfrac{1+\sqrt{5}}{2}$.

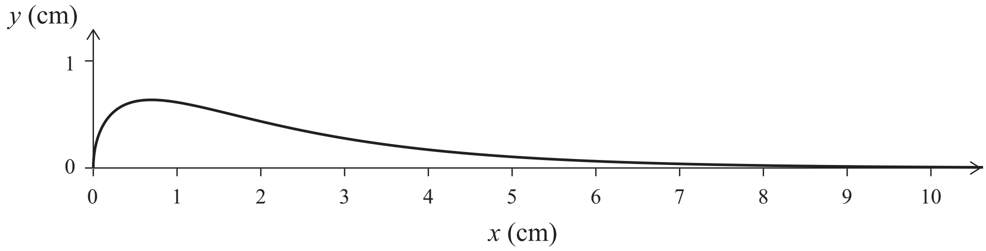

Figure 2: A mathematical model for a drop as a volume of revolution.

(a) Where is the modelled drop widest, and how wide is it there?

(b) The drop changes shape near B in Figure 1, where the concavity of the revolved function is zero.

Use $\dfrac{d^2y}{dx^2} = \sqrt{\phi}\,\dfrac{(e^{2x} - 6e^{x} + 4)}{y^2 e^{4x}}$ to find the exact x coordinate of B.

(c) The volume formed by rotating a curve $y = f(x)$ between $x = a$ and $x = b$ is given by $\int_a^b \pi y^2 \, dx$.

Show that the volume of the drop between $x = 0$ and $x = \ln(p)$ is $V = \dfrac{\pi\phi}{2}\left(\dfrac{p-1}{p}\right)^2$.

Hence or otherwise, explain why the volume of the drop is never more than some upper limit V_L, no matter how long its tail.

QUESTION TWO (8 Marks)

This question defines four new terms related to functions and their definite integrals over a specified interval.

1 The **inner product** of two continuous functions f and g over the interval $a \leq x \leq b$ is

$$\langle f, g \rangle_a^b = \int_a^b f(x)g(x)\,\mathrm{d}x$$

2 The **norm** of the continuous function f over the interval $a \leq x \leq b$ is

$$\|f\|_a^b = \sqrt{\langle f, f \rangle_a^b}$$

3 The **angle** θ **between two functions** f and g over the interval $a \leq x \leq b$ is given by

$$\cos\theta = \frac{\langle f, g \rangle_a^b}{\|f\|_a^b \cdot \|g\|_a^b} \ \textit{where} \ \|f\|_a^b \neq 0,\ \|g\|_a^b \neq 0$$

4 Two functions f and g are **orthogonal** over the interval $a \leq x \leq b$ if the angle between them is $\frac{\pi}{2}$

(a) Find the exact values of k for which $f(x) = kx + 1$ and $g(x) = x + k$ are orthogonal over $0 \leq x \leq 1$.

(b) Consider the functions $p(x) = 3x - 4$ and $q(x) = 9x - 5$ over $0 \leq x \leq 1$.

Find the exact angle between the two functions.

(c) For what positive integers n and m are $\sin(nx)$ and $\sin(mx)$ orthogonal over $0 \leq x \leq 2\pi$?

QUESTION THREE (8 Marks)

(a) A function f is **even** if $f(-x) = f(x)$ for all x in its domain.
A function f is **odd** if $f(-x) = -f(x)$ for all x in its domain.

(i) Recall that a polynomial is a function in the form $p(x) = a_0x^0 + a_1x^1 + \ldots + a_nx^n$.

Describe which polynomials are even, and which are odd, and which are neither.

(ii) Suppose that g is any even differentiable function defined for all real numbers (not necessarily a polynomial).

Use the limit definition of the derivative to prove that $\dfrac{\mathrm{d}g}{\mathrm{d}x}$ is an odd function.

(b) Suppose $y = \mathrm{e}^{-x}\sin(kx)$, where k is a non-zero constant.

Find the values of k for which $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} = Cy$, and hence find the value of C.

QUESTION FOUR (8 Marks)

(a) Find all the points which satisfy $z^n = z$, where z is a complex number, and n is a whole number where $2 \leq n \leq 9$.

How many different solutions are there altogether?

(b) (i) The relativistic rocket equation is below.

$$\frac{m_0}{m_1} = \left(\frac{1 + \frac{\Delta v}{\mathrm{c}}}{1 - \frac{\Delta v}{\mathrm{c}}} \right)^{\frac{\mathrm{c}}{2\mathrm{u}}}$$

Show that this equation rearranges to $\Delta v = \mathrm{c} \cdot \tanh\left(\frac{\mathrm{u}}{\mathrm{c}} \ln\left(\frac{m_0}{m_1} \right) \right)$

where the hyperbolic tangent function is $\tanh(x) = \frac{\mathrm{e}^{2x} - 1}{\mathrm{e}^{2x} + 1}$.

(ii) The relativistic rocket equation is derived from the following differential equation, where u and c are constants.

$$\frac{\mathrm{d}M}{\mathrm{d}v} = \frac{-M}{\mathrm{u}\left(1 - \frac{v^2}{\mathrm{c}^2} \right)}$$

Show that $\ln M = \frac{-\mathrm{c}}{2\mathrm{u}} \ln\left(\frac{1 + \frac{v}{\mathrm{c}}}{1 - \frac{v}{\mathrm{c}}} \right)$ is a solution of this differential equation.

QUESTION FIVE (8 Marks)

(a) A goat is tethered with a rope of length R, to a point P, due west of the end of a fence, which runs due north. The region of grass the goat can reach is shown in Figure 3.

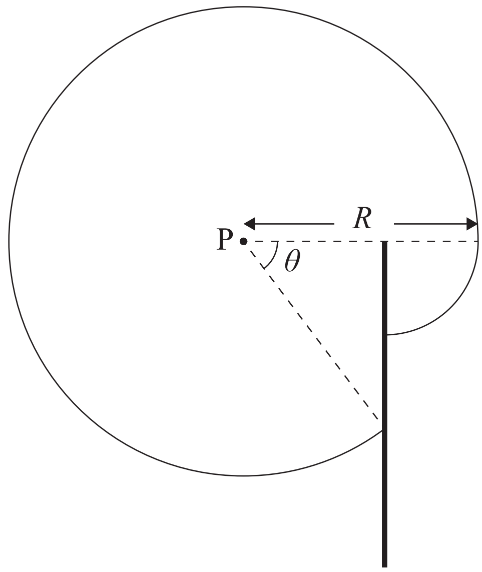

Figure 3: The area reached by a goat tethered near the end of a fence.

The following equation gives the area the goat can reach, A, as a function of the angle θ shown, where $0 \le \theta \le \frac{\pi}{2}$.

$$A(\theta) = \frac{2\pi - \theta}{2} R^2 + \frac{1}{4}\pi R^2 (1 - \cos\theta)^2 + \frac{1}{2} R^2 \sin\theta \cos\theta$$

Find the value of θ where $A(\theta)$ is minimised.

All working must be shown.

(b) Answer ONE of the following options.

EITHER

Consider the following linear inequalities, where a is a positive constant.

$x \le 2a$

$y \le 3a$

$x + y \le 4a$

Draw a **non-linear** objective function $P(x,y)$ for which there are **exactly** two points in the feasible region which maximise the objective function.

Justify your answer carefully.

OR

Figure 4 shows the tasks that need to be completed in painting two small rooms. Estimated durations for the tasks are shown in brackets (in minutes).

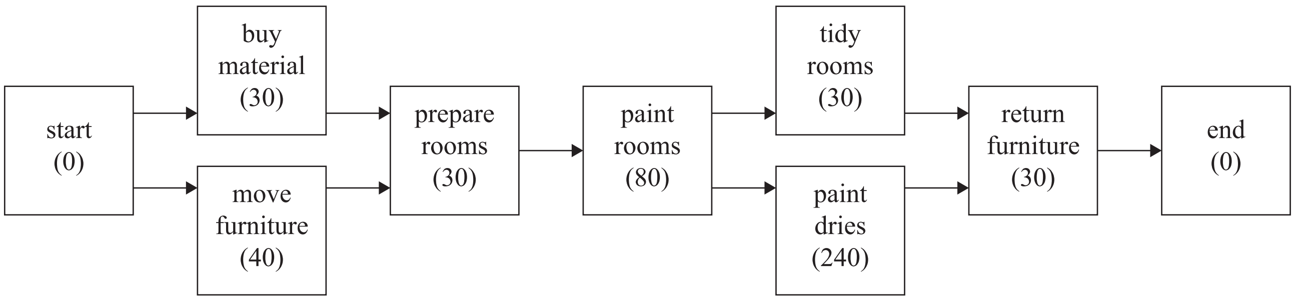

Figure 4: Flow chart and expected durations (in minutes) of tasks for painting two rooms.

The painters realise that they could complete the job faster by working on each room separately, and start a new plan, as shown in Figure 5.

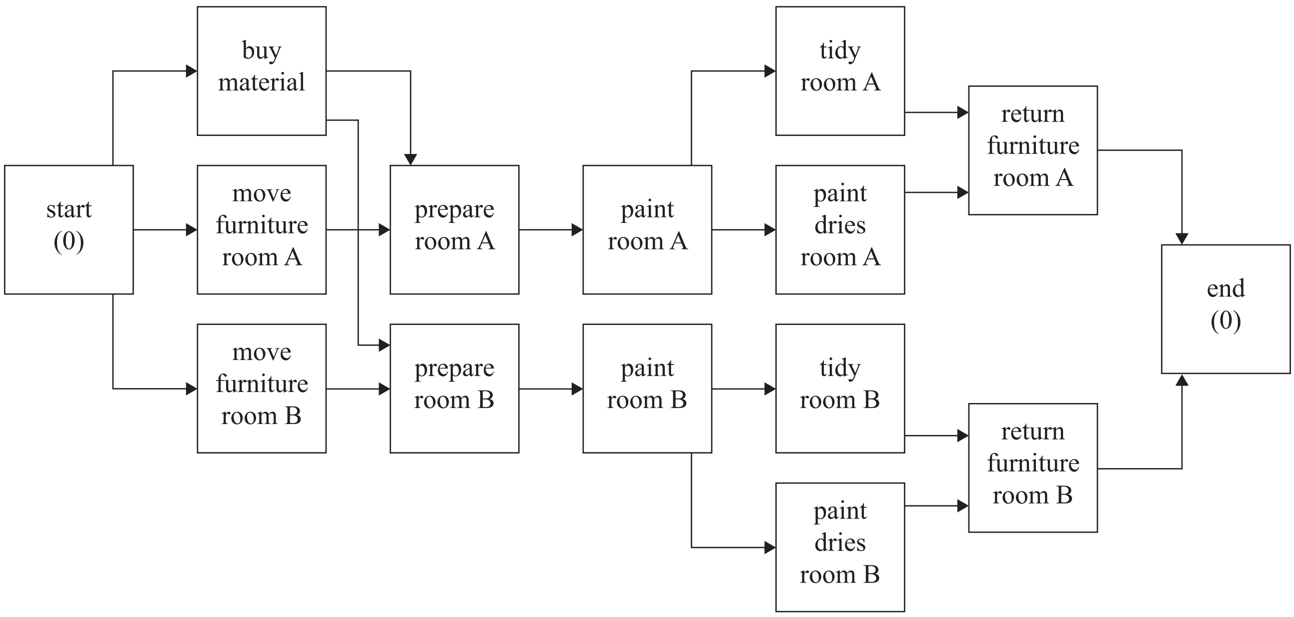

Figure 5: Incomplete flow chart of tasks for painting two rooms.

Discuss how much faster the painters might complete the task, and any real-world limitations that may restrict them.

Figure 5 is reproduced on page 27 of the answer booklet so that you can show any working.

Question Six starts on the following page.

QUESTION SIX (8 Marks)

(a) By considering the expansion of $(\text{cis}\,\theta)^5$, or otherwise, prove both of the following identities.

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

(b) Answer ONE of the following options.

EITHER

A teacher sets 99 homework questions for her Calculus class each week, of three different types: easy, difficult, and impossible. The number of questions of each type, given in week *n,* are represented by x_n, y_n, and z_n respectively.

The teacher uses the following **system of linear equations** to vary the number of questions of each type given each week.

$$x_{n+1} = 0.8x_n + 0.7y_n + 0.6z_n$$

$$y_{n+1} = 0.1x_n + 0.2y_n + 0.4z_n$$

$$z_{n+1} = 0.1x_n + 0.1y_n$$

Her class notice that the number of questions of each type stabilises after several weeks. That is, in the long run they notice that $x_{n+1} = x_n$, $y_{n+1} = y_n$, and $z_{n+1} = z_n$.

How many questions of each type will the teacher give each week once the numbers stabilise?

OR

Use your knowledge of **ellipses** to sketch all points in the complex plane satisfying the following inequalities, where k is a positive constant.

$$k \le |z + \mathrm{i}| + |z - \mathrm{i}| \le 2k$$

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