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93202Q

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NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

Scholarship 2014
Calculus

9.30 am Wednesday 19 November 2014
Time allowed: Three hours
Total marks: 40

QUESTION BOOKLET

There are five questions in this booklet. Answer ALL FIVE questions, choosing ONE option from part (b) of each question.

Write your answers in Answer Booklet 93202A.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Start your answer to each question on a new page. Carefully number each question.

Check that this booklet has pages 2–7 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

This examination consists of FIVE questions.

Answer all FIVE questions, choosing ONE option from part (b) of each question.

QUESTION ONE (8 Marks)

(a) Find all solutions of $\left(7+4\sqrt{3}\right)^{t^2-5t+5}+\left(7-4\sqrt{3}\right)^{t^2-5t+5}=14$.

Give the solutions in exact form.

(b) *Answer ONE of the following options.*

EITHER

Find all solutions of $\left[\ln\left(\sin^{-1}\mathrm{e}^{x}\right)\right]^{5}=\ln\left(\sin^{-1}\mathrm{e}^{x}\right)$, where x is real.

Give the solutions in exact form.

OR

A hospital has 150 rooms that it intends to upgrade into double or triple rooms. The hospital can spend no more than $1 000 000 in upgrading the rooms. The cost to upgrade each double room is $6400, and the cost to upgrade each triple room is $8000.

There are at least 4 requests for double rooms for every 3 requests for triple rooms, and the hospital decided to consider this in its planning. The daily income on a double room is $1200 per room, and the daily income on a triple room is $1800 per room.

Find the combination of double and triple rooms that maximises the potential daily income, assuming total occupancy of the beds.

Comment on any limitation(s) with your answer, and suggest a way to resolve the matter(s).

QUESTION TWO (8 Marks)

(a) Consider the equation $x^4 - 2kx^2 + q^2 = 0$, where k and q are real numbers.

Describe how the nature of the roots of this equation depends on k and q.
Your answer should cover complex and repeated roots.

(b) *Answer ONE of the following options.*

EITHER

Find exact expressions for the areas of the three labelled regions bounded by the two curves $y = 9\operatorname{cosec}^2 x$ and $y = 16\sin^2 x$ between $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$, shown in Figure 1.

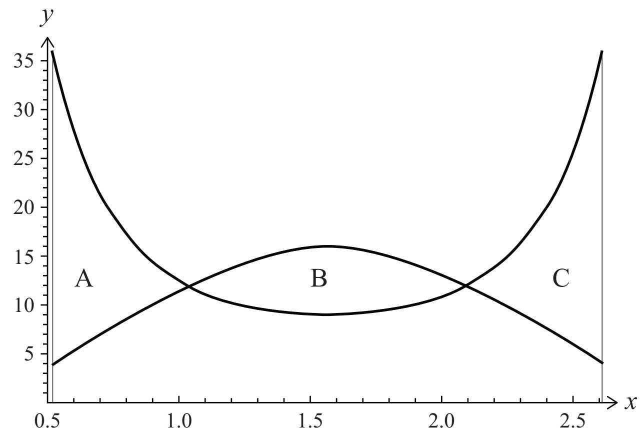


Figure 1: The curves $y = 9\operatorname{cosec}^2 x$ and $y = 16\sin^2 x$ between $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

OR

The following system of linear equations with variables x, y and z contains a constant c.

$$(c + 2)x + (c - 2)z = 0$$
$$(c + 3)x + (c - 3)y = 0$$
$$(c + 5)y + (c - 5)z = 0$$

Investigate how the solutions of the linear system depend on the value of c.

QUESTION THREE (8 Marks)

(a) A narrow rod 12 cm long is made of a material of varying density.

The density of the rod at a point x cm from one end A of the rod is given by $\rho(x) = bx^r(12 - x)$ in the interval $0 \leq x \leq 12$, where b and r are positive constants.

The centre of gravity of the rod is at the point c cm from A such that $\int_0^{12} \rho(x)(x - c)\,\mathrm{d}x = 0$.

Find the centre of gravity of the rod, in terms of r.

(b) *Answer ONE of the following options.*

EITHER

A family of functions is built from two functions $f(x)$ and $g(x)$, with a new function $h_p(x)$ defined for each value of p, $0 \leq p \leq 1$. These functions are shown in Figure 2.

$f(x) = 2 + \sin x$

$g(x) = 26 + \sin x$

$h_p(x) = [f(x)]^{1-p}\,[g(x)]^p$

The function $S(p)$ represents the difference between the maximum and minimum values of $h_p(x)$.

Find the exact value of p that maximises $S(p)$.

Note that if a is a constant, $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{a}^x) = \ln \mathrm{a} \cdot \mathrm{a}^x$.

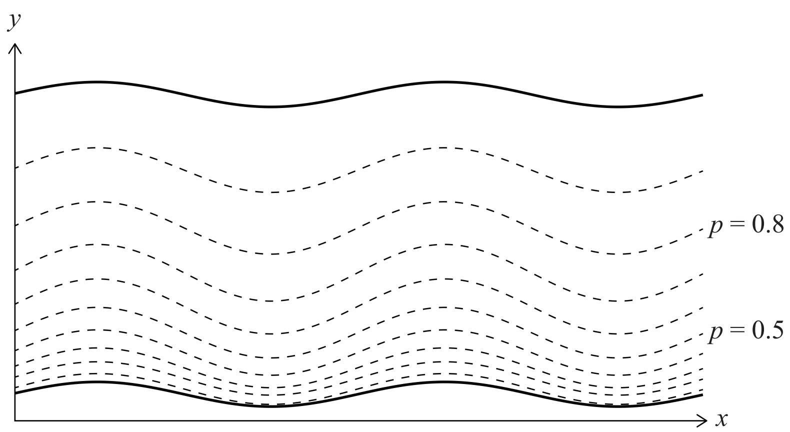


Figure 2: Graphs of $y = f(x)$, $y = g(x)$, and $y = h_p(x)$ as dashed lines with various values of p from 0.1 to 0.9.

OR

A toy developer is designing a new product to be manufactured in its factory. The design team has identified 11 activities and their precedence relationships. The project needs to be completed in 38 weeks.

Find the critical path, crash the project to fit the 38-week deadline, state the changes that you make and their effect, and find the most economical crashing cost to the clients.

Note: a network diagram with project duration times is in your answer booklet on pages 26 and 27.

Table 1: Activity precedence and duration details for new product manufacture

Activity	Description	Precedence	Duration (weeks)
A	Design the product		4
B	Design the manufacturing process	A	7
C	Purchase materials	A	3
D	Purchase manufacturing equipment	B	5
E	Install manufacturing equipment	D	15
F	Receive materials	C	6
G	Pilot production run	E, F	2
H	Evaluate product design	G	2
I	Evaluate manufacturing process performance	G	3
J	Obtain client approval	H, I	4
K	Make alterations and commence manufacturing	J	3

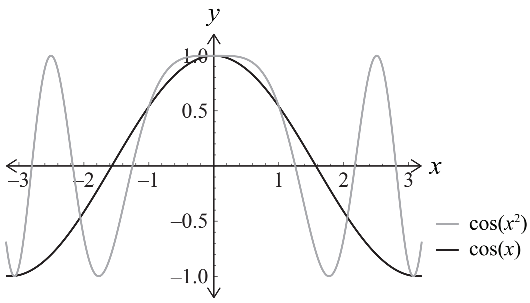
Table 2: Activity crash costs for new product manufacture

Activity	Duration (weeks)	Normal cost ($)	Crash time (weeks)	Crash cost ($)	Max. weeks of reduction	Increased cost per week saved ($)
A	4	12 000	3	16 000	1	4 000
B	7	30 000	5	35 000	2	2 500
C	3	6 000	3	6 000	0	0
D	5	24 000	4	27 000	1	3 000
E	15	65 000	13	80 000	2	7 500
F	6	5 000	4	8 000	2	1 500
G	2	6 000	2	6 000	0	0
H	2	4 000	2	4 000	0	0
I	3	4 000	2	6 000	1	2 000
J	4	3 000	2	5 400	2	1 200
K	3	6 000	2	8 000	1	2 000

QUESTION FOUR (8 Marks)

(a) Find the exact values of x for which $\text{cis}(x^2) = \text{cis}\,x$, between $-\pi$ and π.

Figure 3 may be useful.



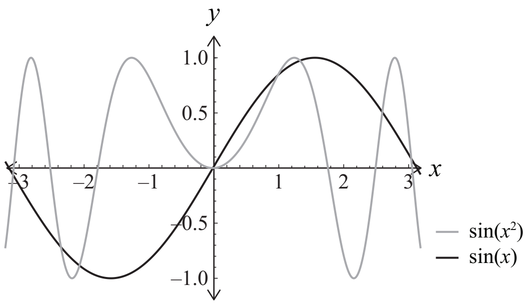


Figure 3: Graphs of the real and imaginary parts of $\text{cis}(x^2)$ and $\text{cis}\,x$ with $-\pi \leq x \leq \pi$.

(b) *Answer ONE of the following options.*

EITHER

The rule for the derivative of a triple product is given below.

$$(fgh)' = f'gh + fg'h + fgh'$$

Using this rule, or otherwise, find the values of the coefficients A to J in the following rule for the *third* derivative of a triple product.

$$(uvw)''' = Au'''vw + Buv'''w + Cuvw''' + Du''v'w + Eu''vw' + Fu'v''w + Guv''w' + Hu'vw'' + Iuv'w'' + Ju'v'w'$$

OR

There are many integer solutions to the equation $\binom{n}{r} = \binom{n+1}{r-1}$, including $n = r = 1$.

Find an expression for n in terms of r, and hence find another of the integer solutions.

QUESTION FIVE (8 Marks)

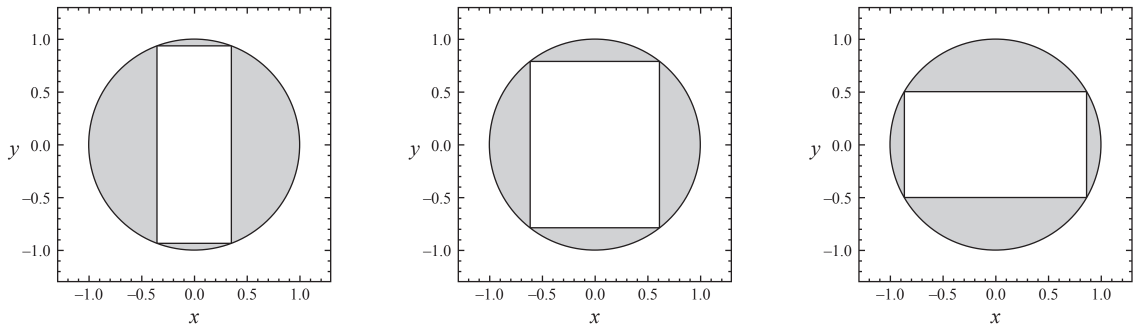
(a) Show that $y = \mathrm{e}^{\mathrm{e}^{cx}}$ is a solution of the differential equation $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = c^2 \cdot y \cdot \ln y \cdot (1 + \ln y)$.

(b) *Answer ONE of the following options.*

EITHER

Consider the function $F(x, y) = (x^2 - A)(y^2 - 1 + A)\sqrt{1 - x^2 - y^2}$, where $0 < A < 1$.
The regions where $F(x,y) \le 0$ for various values of A are shaded in Figure 4.

Describe how the boundaries of the shaded regions arise from the solution of $F(x,y) = 0$, **and** hence, or otherwise, find the exact values of A for which the shaded region is half the area of the circle.



$A = 0.125$ $A = 0.375$ $A = 0.75$

Figure 4: Regions showing the positive values of $F(x,y)$ shaded grey, for various values of A.

OR

An ellipse with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is enclosed by the hyperbolas given by $xy = 1$ and $xy = -1$.

Determine the largest area of an ellipse enclosed by the hyperbolas.

Note that the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $A = \pi ab$.