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# Scholarship 2015
# Calculus

2.00 p.m. Tuesday 17 November 2015
Time allowed: Three hours
Total marks: 40

## QUESTION BOOKLET

There are five questions in this booklet. Answer ALL FIVE questions, choosing ONE option from part (b) of Question Four.

Write your answers in Answer Booklet 93202A.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–8 in the correct order and that none of these pages is blank.

**YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.**

**QUESTION ONE**

(a) A solid of revolution is a three-dimensional figure formed by revolving a plane area around a given axis.

The surface area of a solid of revolution, which has been revolved 360° around the $x$-axis, is given by:

$$\text{surface of revolution } = \int_a^b \left( 2\pi\, f(x)\sqrt{1+\left[f'(x)\right]^2}\, \right) dx$$

Find the area of the surface of revolution obtained when the graph of $f(x) = x^3 + \dfrac{1}{12x}$, from $x = 1$ to $x = 3$, is revolved 360° around the $x$-axis.

(b) Determine all differentiable functions of the form $y = f(x)$ which have the properties:

$$f'(x) = \left(f(x)\right)^3 \text{ and } f(0) = 2.$$

(c) A tank contains 200 litres of brine (solution of salt in water). Initially the concentration is 0.5 kg of salt per litre. Brine containing 0.8 kg of salt per litre runs into the tank at a rate of 6 litres per minute. The mixture is kept thoroughly mixed and is running out at the same rate.

Find how long it takes for the amount of salt in the tank to be 130 kg.

**QUESTION TWO**

(a) Solve the following simultaneous equations for real values of $x$ and $y$:

$$\begin{cases} 9^{2x+y} - 9^{x} \times 3^{y} = 6 \\ \log_{x+1}(y+3) + \log_{x+1}(y+x+4) = 3 \end{cases}$$

(b) A car is travelling at night along a road shaped like a parabola, with its vertex at the origin. The car starts at a point 100 m west and 100 m north of the origin. There is a statue located 100 m east and 50 m north of the origin.

At what point on the road will the car's headlights illuminate the statue?

(c) The rate of spread of a rumour at a particular school is proportional to both the number of students who know a rumour, $S$, and the number of students who do not.

If $N$ is the total number of students in the school, then $\dfrac{\mathrm{d}S}{\mathrm{d}t} = kS(N-S)$.

Initially two students knew the rumour.

Show that the number of students who know the rumour at time $t$ is $S(t) = \dfrac{N}{1+\frac{1}{2}\mathrm{e}^{-kNt}(N-2)}$.

**QUESTION THREE**

(a) Prove that if $z = \cos\theta + \mathrm{i}\sin\theta$, then $z^n + \dfrac{1}{z^n} = 2\cos n\theta$.

Hence, or otherwise, prove that $\cos^6\theta = \dfrac{1}{32}\left(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right)$.

(b) An arithmetic sequence is a sequence of numbers such that the difference between the consecutive terms is constant.

The following numbers form three consecutive terms of an arithmetic sequence, for some real $x$ and $y$:

$\log 2,\ \log(2\sin x - 1),\ \log(1 - y)$

Find the range of possible values for $y$.

(c) Prove that $\dfrac{4\cos^2 2x - 4\cos^2 x + 3\sin^2 x}{4\cos^2\left(\dfrac{5\pi}{2} - x\right) - \sin^2 2(x - \pi)} = \dfrac{8\cos 2x + 1}{2(\cos 2x - 1)}$

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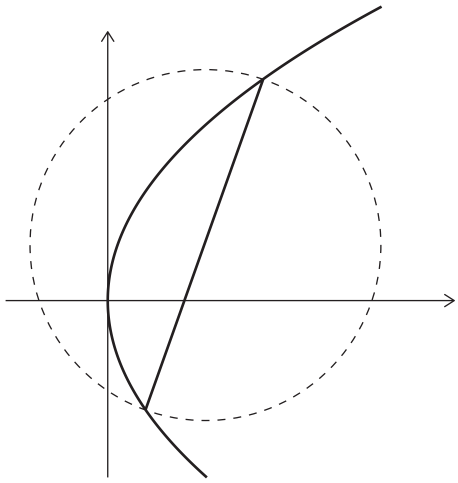
**QUESTION FOUR**

(a) Show that the equation $3z^3 + (2 - 3a\mathrm{i})z^2 + (6 + 2b\mathrm{i})z + 4 = 0$ (where both $a$ and $b$ are real numbers) has exactly one real root, and find this root.

(b) *Answer ONE of the following options.*

**EITHER**

Prove that the directrix is tangent to the circles that are drawn on a focal chord of a parabola as diameter.



*Note: A focal chord is a line segment that passes through the focus of a parabola and has its endpoints on the parabola.*

**OR**

A factory manufactures cake decorations.

The factory makes three different kinds of packs: red, white, and blue.

Each pack contains three different types of decorations: type A, B, and C.

Each red pack has 6 type A decorations, 3 type B decorations, and 2 type C decorations.

Each white pack has 2 type A decorations, 4 type B decorations, and 4 type C decorations.

Each blue pack has 2 type A decorations, 5 type B decorations, and 2 type C decorations.

The factory makes $x$ red, $y$ white and $z$ blue packs every hour.

The maximum number of each type of decoration available is 500 type A, 400 type B, and 300 type C every hour.

The factory must pack at least 1000 decorations every hour.

The factory must pack more type B decorations than type A decorations every hour.

(i) Write, as inequalities, the 5 constraints in addition to $x \geq 0, y \geq 0, z \geq 0$.

Simplify your answers.

(ii) Given that every hour the factory produces 100 packs in total, identify the feasible region on the diagram on page 27 in the answer booklet (also shown below).

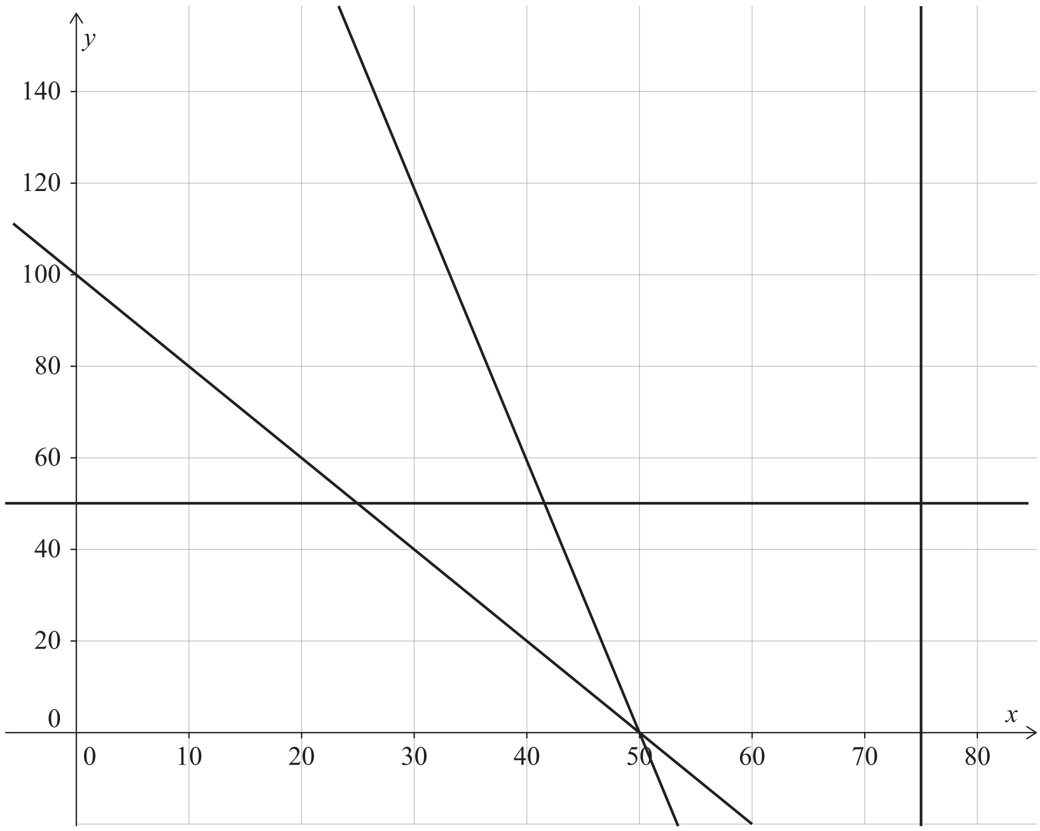
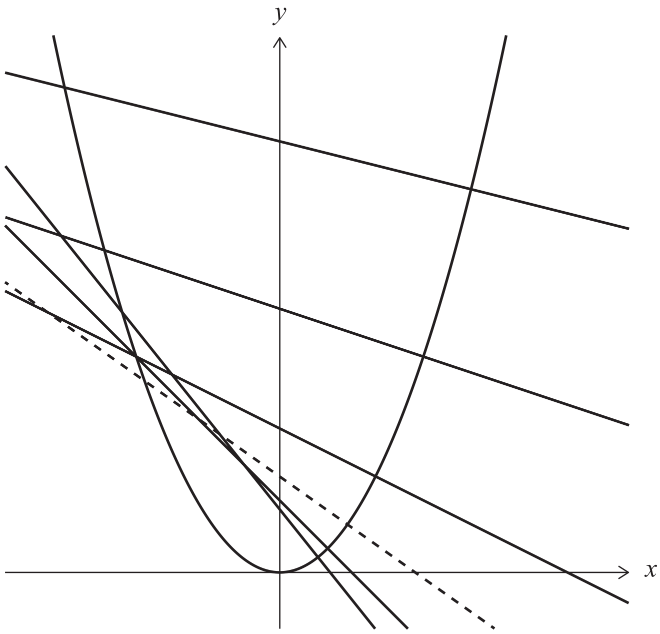



Diagram from page 27 of the answer booklet.

*Please use the diagram in the answer booklet for your answer.*

**QUESTION FIVE**

The curve in the figure below is the parabola $y = kx^2$, where $k > 0$.




Several normal lines to this parabola are also shown. Consider the points in the first quadrant from which the normal lines are drawn. Notice that as the $x$-coordinate of the point gets smaller, the $y$-coordinate of the intersection of the normal with the other arm of the parabola also decreases until it reaches a minimum, and then it increases. The normal line with the minimum $y$-coordinate is dotted.

(a) Show that the equation of the normal to the parabola at a point $(x_0, y_0)$ is $y = \frac{-1}{2kx_0}x + kx_0^2 + \frac{1}{2k}$ .

(b) Show that the equation of the normal line with the minimum $y$-coordinate is $y = \frac{-\sqrt{2}}{2}x + \frac{1}{k}$.

(c) Find the equation of the normal that produces the smallest area between itself and the parabola, and find this area.