93202Q

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NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2016
Calculus

9.30 a.m. Friday 25 November 2016
Time allowed: Three hours
Total marks: 40

QUESTION BOOKLET

There are five questions in this booklet. Answer ALL FIVE questions, choosing ONE option from part (b) of Question Five.

Write your answers in Answer Booklet 93202A.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–5 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

QUESTION ONE

(a) Given that (x,y) is any point on the circle $(x-2)^2 + y^2 = 3$, find the maximum value of $\frac{y}{x}$.
You do **not** need to show that your answer is a maximum.

(b) For real numbers $x > -1$,

$$f(x) = x^2 \ln(x+1).$$

(i) Find the second derivative of $f(x)$, $\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}$, then find the value of this derivative at $x = 0$, i.e. $f^{(2)}(0)$.

(ii) Find the value of the 2016th derivative of $f(x)$ at $x = 0$,

$$\left(\frac{\mathrm{d}^{2016} f(x)}{\mathrm{d}x^{2016}}\right)_{x=0}, \text{ i.e. } f^{(2016)}(0)$$

Write your answer in factorial form.

QUESTION TWO

(a) Find the following integral, writing your answer in exact form and showing all your working:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^5 x + \cos^5 x)\,\mathrm{d}x$$

(b) Consider $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{\sin x}\,\mathrm{d}x$, where $n \geq 0$.

Show that $I_n - I_{n-1} = \frac{2(-1)^{n-1}}{2n-1}$.

(c) Given that $\log_{(\tan\theta + \cot\theta)}(\cos\theta) = k$, where k is a real number, find an expression for

$$\log_{\tan\theta}(\sin\theta)$$

in terms of k.

QUESTION THREE

The Integration by Parts formula is: $\int f \cdot g' \mathrm{d}x = f \cdot g - \int f' \cdot g \, \mathrm{d}x$

For definite integrals, it becomes: $\int_a^b f \cdot g' \mathrm{d}x = \left[f \cdot g \right]_a^b - \int_a^b f' \cdot g \, \mathrm{d}x$

You may use these Integration by Parts formulae in this question.

(a) A function $f(x)$, where x is a real number, is defined implicitly by the following formula:

$$f(x) = x - \int_0^{\frac{\pi}{2}} f(x) \sin x \, \mathrm{d}x$$

Find the explicit expression for $f(x)$ in its simplest form.

(b) A curve passing through the point (1,1) has the property that, at each point P (x,y) on the curve, the gradient of the curve is $x - 2y$, that is, $\frac{\mathrm{d}y}{\mathrm{d}x} = x - 2y$.

(i) Show that $\frac{\mathrm{d}\left(\mathrm{e}^{2x} y\right)}{\mathrm{d}x} = x\mathrm{e}^{2x}$.

(ii) Hence, or otherwise, find the equation of the curve.

QUESTION FOUR

(a) P is a point on the right branch of the hyperbola $\frac{x^2}{4} - \frac{y^2}{36} = 1$.

θ is the angle between the two asymptotes, l_1 and l_2, of the hyperbola, as shown on the diagram below.

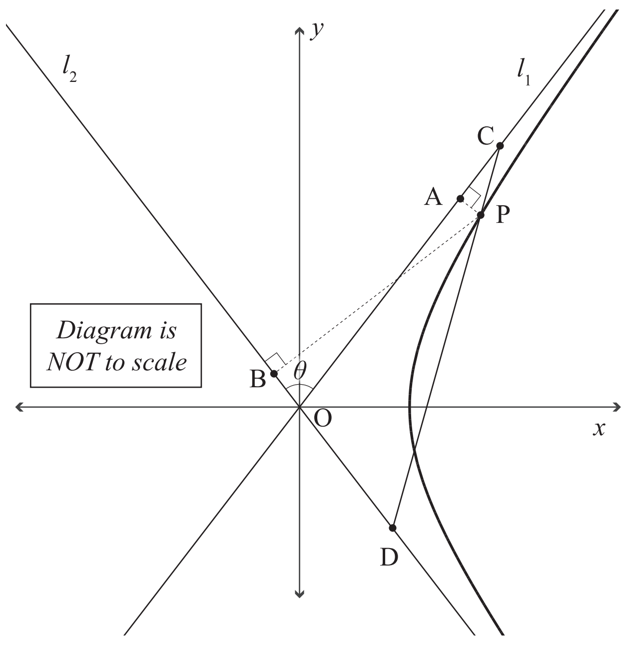
Show that $\sin\theta = 0.6$.

(b) From P, lines are drawn perpendicular to the asymptotes. The points where the perpendiculars meet the asymptotes are A and B, as shown on the diagram.

Show that the product of the lengths of the two segments PA and PB is constant, and state its value.

(c) Consider a case where P is located in the first quadrant. Through P, draw another line CD, where C is on line l_1, D is on line l_2, and P is between C and D, such that CP : PD = 1 : λ.

Find the value of λ, which will minimise the area of the triangle COD, and find this area.



QUESTION FIVE

(a) Find the roots of the equation, $z^{11} = 1$. Hence, or otherwise, show that

$$\cos\left(\frac{2\pi}{11}\right)+\cos\left(\frac{4\pi}{11}\right)+\cos\left(\frac{6\pi}{11}\right)+\cos\left(\frac{8\pi}{11}\right)+\cos\left(\frac{10\pi}{11}\right)=-\frac{1}{2}$$

(b) Answer ONE of the following options.

EITHER:

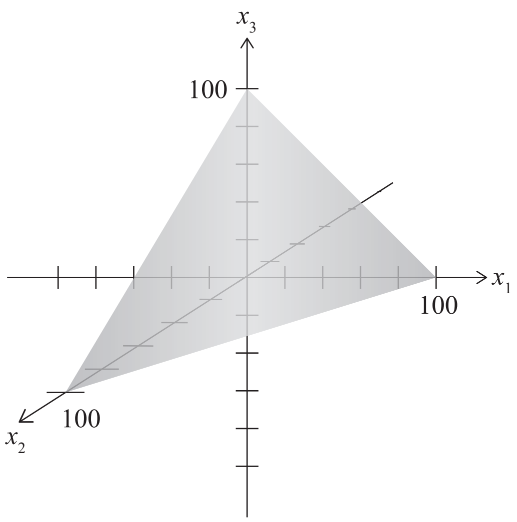
A student is to sit an examination. The questions are divided into three groups. The student may answer any question from any group so long as the total number of questions answered does not exceed 100. The groups are characterised as follows:

- Group 1 – easy, worth four marks each, and will take an average time of two minutes per question to answer.
- Group 2 – moderate difficulty, worth five marks each, and will take an average time of three minutes per question to answer.
- Group 3 – the most difficult, worth six marks each, and will take an average time of four minutes per question to answer.

The total time available to the student is 3 ½ hours. The questions in groups 1 and 2 are the most mechanical and the student can tolerate only 2 ½ hours of this kind of work before losing motivation.

One of the constraints is $x_1 + x_2 + x_3 \leq 100$, where x_1, x_2, and x_3 represent the number of questions answered from Groups 1, 2, and 3, respectively.

Graphically, this constraint can be represented as a plane in three dimensions. It defines a region of solutions to consider, lying between the plane and the origin, including the origin and the plane.

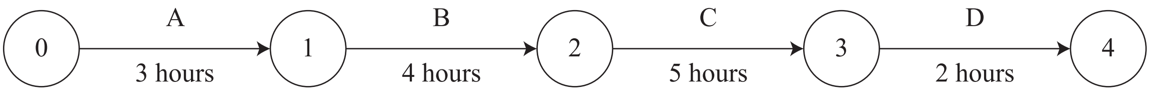


This graph is repeated on pages 26 and 27 of your answer booklet.

Write down the remaining constraints and the objective function.

What combination of questions should the student answer for a maximum grade, assuming all answered questions are correct?

OR:



The diagram above relates to four operations, A, B, C, and D that are performed in series, with the time required to complete each operation also shown.

However, B can be started when A is ⅓ completed, but B cannot be completed until one hour after A is completed. C can be started when B is ½ completed, but C cannot be more than ½ completed until one hour after B is completed. D can begin after A is ⅔ complete but can be no more than ½ completed until B is complete. D can be completed exactly when C is ⅗ complete.

Design a network that incorporates these constraints, and identify the critical path.